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XIX. *Note on Potentials Derived from Axial Values in Electron Optics.*

By FEODORA BERZ, L. ès Sc.,
Mullard Electronic Research Laboratories †.

[Received October 31, 1949.]

SYNOPSIS.

The determination of electrode systems which produce a precalculated distribution along the optic axis is examined. It is shown that an analytic potential distribution is completely determined when its values along any arbitrarily small interval on the axis are exactly known. On the other hand if, as usual, the value of the potential along a finite stretch of the axis is prescribed, within arbitrarily small but finite limits, this is compatible with an infinity of different potential distributions, which can be constructed without any singularity at finite distances, and which can assume prescribed values at any number of points outside the axis of symmetry.

§ 1. INTRODUCTION.

THE determination of electrode systems to produce a precalculated potential distribution along the optic axis is a standard problem of electron optics. By a well-known theorem, the potential on the axis uniquely determines the distribution outside it. This theorem, though very often quoted, is not always properly understood, especially in its practical implications. The reason is that in this extrapolation problem the difference between the rigorous, mathematical, and the everyday or practical concept of a function, which is often overlooked in physical problems, has far reaching consequences.

In order to make this clear, it will be shown in this paper, first that if the concept of function is used in the *mathematical* sense, the potential function in the plane is determined by its values on a stretch smaller than any prescribed length (its values along the whole axis being either redundant, or leading to non-analytic solutions). Second, it will be shown that if the concept of the "given function" is used in its *practical* meaning, *i. e.* if the potential on the axis is known within finite but arbitrarily small limits, the potential problem admits an infinity of solutions.

The chief practical consequence of these results is that in many electron optical problems, where the exact realization of a precalculated potential distribution on the axis leads to inconvenient or even impossible shapes, very small variations of the data, which do not appreciably influence the electron trajectories, may result in electrode systems far more convenient for practical application.

† Communicated by Dr. D. Gabor.

For convenience we shall limit ourselves to plane electrostatic fields, with potential distributions symmetrical with respect to the OX axis. The case of rotational symmetry is more complicated, but the general results obtained can be transferred without alteration, as will be briefly summed up in §4, at the end of this paper.

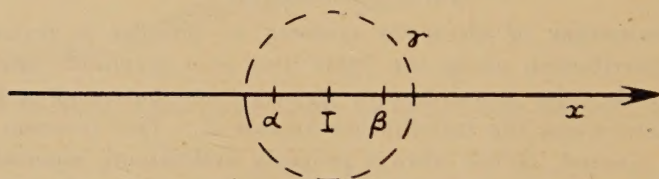
§ 2. RIGOROUS MATHEMATICAL SOLUTION—THEOREM OF UNIQUENESS.

Consider a potential distribution $V(x, y)$ which can be expounded in Taylor series in the neighbourhood of any given point, with the possible exception of some isolated singularities. Such distribution will be called "analytic", by an extension of the theory of functions of a complex variable.

In the absence of space charge, case which will only be considered here, the function $V(x, y)$ must satisfy Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Fig. 1.



If $v(x)$ is the potential distribution along the OX axis, the solution in the whole plane is

$$V(x, y) = \text{Re} v(x + iy), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\text{Re} v(x + iy)$ stands for the real part of the function $v(x + iy)$.

Another well-known formula giving $V(x, y)$ as a power series, and valid in the neighbourhood of the OX axis, is

$$V(x, y) = v(x) - \frac{y^2}{2!} \cdot \frac{d^2 v}{dx^2} + \frac{y^4}{4!} \cdot \frac{d^4 v}{dx^4} + \dots + \frac{(-1)^n y^{2n}}{2n!} \cdot \frac{d^{2n} v}{dx^{2n}} + \dots \quad (3)$$

Formula (3) can be derived from formula (2), by expanding it in a Taylor series.

Theorem of uniqueness: if two potential functions assume the same value along any arbitrarily small interval of the axis of symmetry, they are identical.

The proof may be directly obtained from the corresponding theorem concerning the analytic functions of a complex variable.

Let $V(x, y)$ and $V^*(x, y)$ be two potential distributions which assume the same value along a small portion $\alpha\beta$ of the OX axis (fig. 1), and let $v(x+iy)$ and $v^*(x+iy)$ be the corresponding complex functions defined above.

$$\begin{aligned} V(x, 0) &= v(x), & V(x, y) &= \text{Re}v(x+iy), \\ V^*(x, 0) &= v^*(x), & V^*(x, y) &= \text{Re}v^*(x+iy). \end{aligned}$$

Consider now the analytic function $\phi(x+iy)$ such that

$$\phi(x+iy) = v^*(x+iy) - v(x+iy).$$

Expanding $\phi(x+iy)$ in Taylor's series, with a point I inside $\alpha\beta$ taken as origin, we have

$$\phi(x+iy) = \phi_i + z \frac{d\phi_i}{dz} + \frac{z^2}{2!} \frac{d^2\phi_i}{dz^2} + \dots + \frac{z^n}{n!} \frac{d^n\phi_i}{dz^n} + \dots, \quad (4)$$

where

$$z = x + iy,$$

and the subscript i refers to values taken at the point I.

Since $\phi(x+iy)$ is an analytic function, the value of its derivatives at any given point does not depend on the direction along which this derivative is taken. Therefore

$$\frac{d\phi_i}{dz} = \frac{\partial\phi_i}{\partial x}, \quad \frac{d^n\phi_i}{dz^n} = \frac{\partial^n\phi_i}{\partial x^n}.$$

On the other hand $\phi(z)$ is equal to zero along $\alpha\beta$ by definition. Hence its derivatives with respect to x at the point I, and therefore also all the terms of the Taylor series (4), will vanish. Thus $\phi(x+iy) = 0$ inside the circle of convergence γ of the series (4), whose centre is I (fig. 1). Using analytic continuation it could similarly be shown that $\phi(x+iy) \equiv 0$ everywhere. Hence

$$v^*(x+iy) = v(x+iy)$$

and

$$V^*(x, y) = V(x, y).$$

Thus an analytic potential distribution is completely determined by its values along an *infinitesimal* portion of the axis of symmetry.

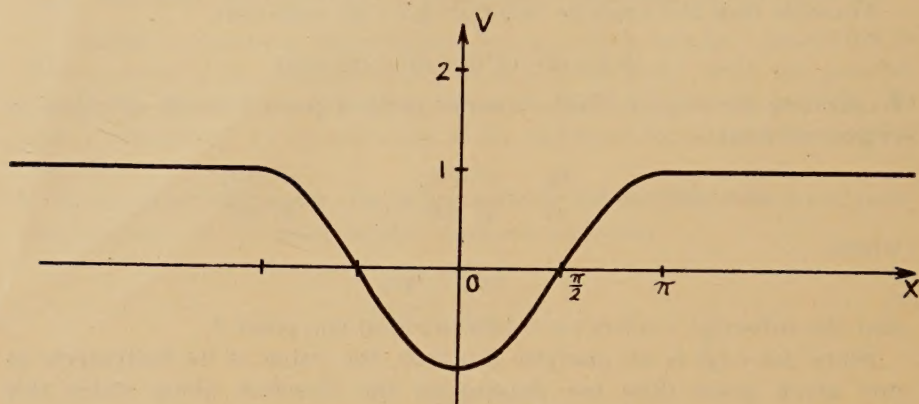
It has to be noted that formulæ (2) (3) tend to give the false impression that the potential distribution along the whole axis OX can be chosen in an arbitrary way. If $v(x+iy)$ is analytic, its values along the whole axis OX are implicitly defined by its values along an infinitesimal portion of this axis. On the other hand if $v(x+iy)$ is not analytic along the axis OX taken as a whole, $V(x, y)$ as defined by (2) or (3) will not be analytic in the neighbourhood of the OX axis.

As an example consider the function $V(x, 0)$ defined by

$$\begin{aligned} V(x, 0) &= -\cos x & \text{for } -\pi \leq x \leq \pi, \\ V(x, 0) &= 1 & \text{for } x > \pi \text{ and } x < -\pi. \end{aligned}$$

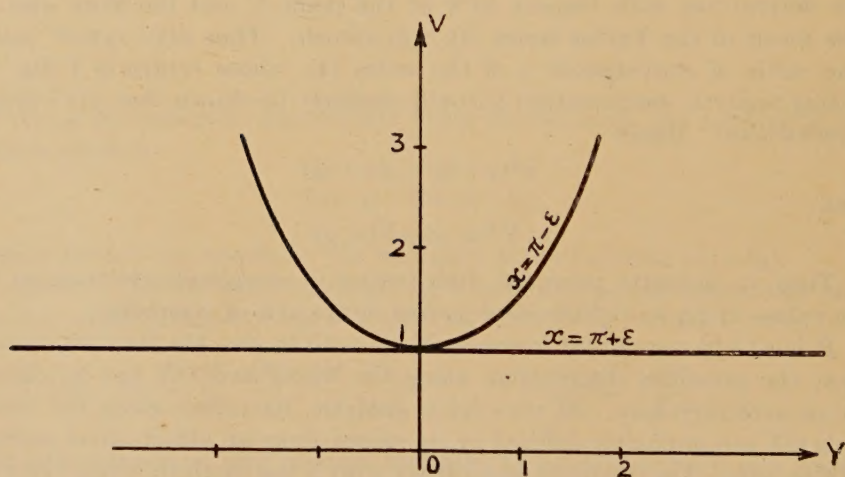
Such a function $V(x, 0)$ is continuous as well as its first derivative (see fig. 2a), but it is not analytic as a whole, on account of the discontinuity of higher derivatives at the points $x = \pm\pi, y = 0$. The potential in the plane, as derived from (2) or (3), is given by

Fig. 2a.



Potential variation along the OX axis. $V(x) = -\cos x$ for $-\pi < x < \pi$; $V(x) = 1$ for $x > \pi$ and $x < -\pi$.

Fig. 2b.



Potential variation along the parallels to the OX axis corresponding to $x = \pi - \epsilon$ and $x = \pi + \epsilon$.

$$\begin{aligned} V(x, y) &= -\cos x \cosh y & \text{for } -\pi \leq x \leq \pi, \\ V(x, y) &= 1 & \text{for } x > \pi \text{ and } x < -\pi. \end{aligned}$$

Such a potential is discontinuous along the lines parallel to oy , for $x = -\pi$ and $x = +\pi$, since for $\epsilon > 0$

$$\lim_{\epsilon \rightarrow 0} V(\pi - \epsilon, y) = \cosh y,$$

$$\lim_{\epsilon \rightarrow 0} V(\pi + \epsilon, y) = 1,$$

and similarly at $x = -\pi$, (fig. 2*b*).

§ 3. APPROXIMATE PRACTICAL SOLUTIONS.

The uniqueness theorem appears to impose very drastic restrictions on the solution of potential problems. It states that potential distributions, even if they are continuous, cannot be arbitrarily prescribed on the axis of symmetry, if an analytic solution is required, and that no alteration of an analytic potential distribution along a finite portion of the axis can be made, without introducing discontinuities, and other very serious singularities.

However these theoretical limitations are of little consequence in practical cases for two reasons. The physicist wants to obtain a given potential distribution along a *finite* portion of the axis of symmetry, and not along the whole axis. Moreover, this potential distribution need be realized only *approximately*. Physical quantities are never measured exactly, and rigorous coincidence is a meaningless concept in physics.

By a theorem of Weierstrass (Goursat 1927), if a function is continuous over a finite interval of the real axis, it can be approximated over this interval with any accuracy by a polynomial †. The following propositions can be derived from this theorem.

Proposition 1.

Any continuous potential distribution, prescribed along a finite section of the axis of symmetry, can be realized with any degree of approximation by an infinity of different potential distributions.

Proposition 2.

These potential distributions can be constructed without any singularity in the plane.

Proposition 3.

These potential distributions can be constructed with prescribed values at any number of points outside the axis of symmetry.

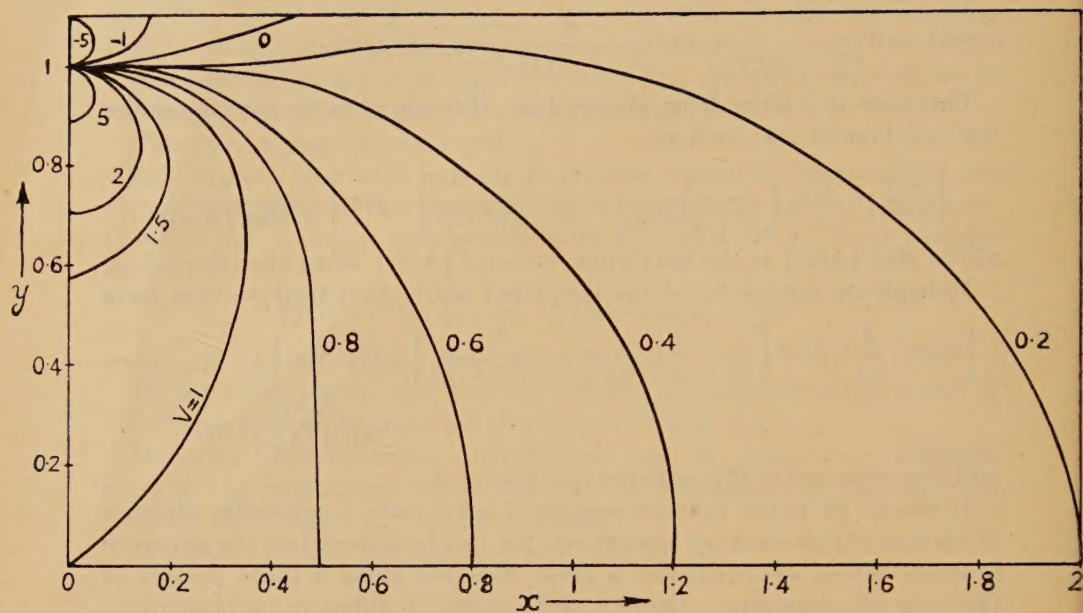
To prove these propositions consider a real polynomial $P(x)$, which approximates the prescribed potential function $v(x) = V(x, 0)$ with the required degree of accuracy over the range $a < x < b$.

If we take

$$V(x, y) = \operatorname{Re} P(z),$$

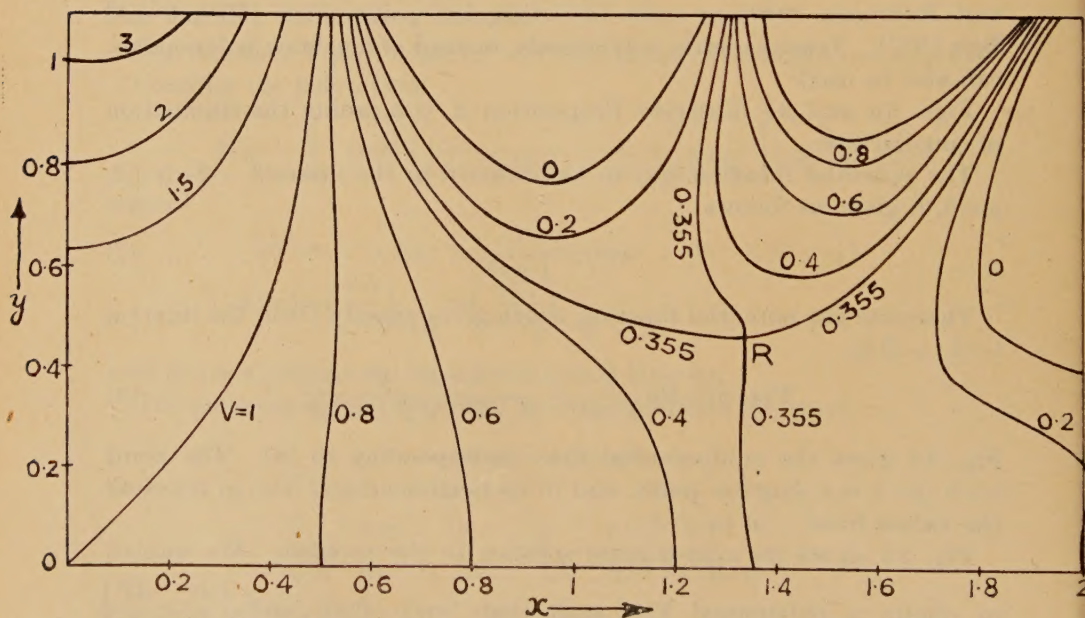
† The approximation of a function by a finite number of terms of its Taylor series is a particular case of Weierstrass' theorem, since it applies only when the function is analytic, and the portion of the real axis over which the approximation is desired lies within the circle of convergence.

Fig. 3a.



Potential distribution which corresponds to $V(x) = \frac{1}{1+x^2}$.

Fig. 3b.



Approximate solution. The potential along the OX axis approximates the function $\frac{1}{1+x^2}$ with an error smaller than 0.015.

has been used†, which takes the same value as $\frac{dv}{dx}$ at the points $x_j = \pm 2 \cos \frac{(2j-1)\pi}{20}$, $j=1, 2, \dots, 5$. (As shown in Appendix I this choice of points leads to a good approximation by interpolation polynomials.)

We then obtain

$$\frac{dP}{dx} = -1.89485x + 2.547087x^3 - 1.477347x^5 + 0.383708x^7 - 0.03636425x^9,$$

and by integration

$$P(x) = 1 - 0.94742x^2 + 0.63677x^4 - 0.246224x^6 + 0.0479635x^8 - 0.00363642x^{10}.$$

The error limits in the interval $-2 < x < 2$, $y=0$ were found to be

$$\left| \frac{dv}{dx} - \frac{dP}{dx} \right| < 0.05, \quad |v(x) - P(x)| < 0.015,$$

where

$$\text{Max} \left| \frac{dv}{dx} \right| = 0.65, \quad \text{Max} |v(x)| = 1.$$

The approximate potential distribution $V^*(x, y)$ is given by

$$V^*(x, y) = \text{Re}P(z) \\ = P(x) - \frac{y^2}{2!} \frac{d^2P}{dx^2} + \frac{y^4}{4!} \frac{d^4P}{dx^4} + \dots - \frac{y^{10}}{10!} \frac{d^{10}P}{dx^{10}}.$$

The last line is an application of formula (3).

Fig. 3b shows the equipotential lines corresponding to $V^*(x, y)$. They are very similar to the lines of fig. 3a in the strip $0 < y < 0.10$. They become very different for values of y larger than 0.30. Of course $V^*(x, y)$ has no singularities, and the branch point R is a point where the electric field is zero.

§ 4. CASE OF ROTATIONAL SYMMETRY.

The same general results are obtained in this case, if formulæ (1) (2) (3) are replaced by

$$\frac{\partial^2 V}{\partial z^2} + \frac{1}{r} \cdot \frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} = 0,$$

$$V(r, z) = \frac{1}{\pi} \int_0^\pi v(z + ir \cos \phi) d\phi,$$

$$V(r, z) = v(z) - \frac{r^2}{2^2} \cdot \frac{d^2 v}{dz^2} + \frac{r^4}{2^2 4^2} \cdot \frac{d^4 v}{dz^4} + \dots + \frac{(-1)^n r^{2n}}{2^2 4^2 \dots (2n)^2} \cdot \frac{d^{2n} v}{dz^{2n}} + \dots, \quad (9)$$

where r and z are cylindrical coordinates.

† If $\left| \frac{dv}{dx} - \frac{dP}{dx} \right| < \epsilon$ we have $|v(x) - P(x)| = \left| \int_0^x \left(\frac{dv}{dx} - \frac{dP}{dx} \right) dz \right| = \epsilon x$. Hence a

good approximation of $\frac{dv}{dx}$ leads to a good approximation of $v(x)$ (if x is not too large), whereas the converse is not true.

Two interesting examples concerning this case are given by N. Plass (1942), who considers the equipotential lines which produce the axial potentials $v(z) = A \exp(-z^2/2)$ (Scherzer's lens of minimum aberration), and $v(z) = A \tanh z$. Plass uses formula (9) where the functions of z are also expanded in Taylor series, and retains only a finite number of terms. N. Plass considers his method to be valid for small values of r only, because he implicitly assumes that the equipotential lines must have approximately the shape of those derived from the rigorous solution. It is seen from the present paper that such an assumption is not necessary, and therefore the results obtained by N. Plass can be extended to large values of r . It has also to be noted that a better approximation is usually obtained when $v(x)$ is expressed by an interpolation polynomial (see Appendix), or orthogonal polynomials, instead of using a finite Taylor series.

ACKNOWLEDGMENTS.

The author is indebted to Dr. D. Gabor of the Imperial College for suggesting this research, and wishes to express appreciation for his helpful interest in this work.

Thanks are also due to the Directors of Mullard Radio Valve Co. for permission to publish this paper.

APPENDIX.

ABOUT THE CHOICE OF INTERPOLATION POINTS.

Known and useful results based on an extremal property of Tchebycheff's polynomials are expounded here, since the works dealing with them are not easily available.

It is well known that if $P_{n-1}(x)$ is the polynomial of degree $n-1$ which takes the same values as $f(x)$ at points x_j ($j=1, 2, \dots, n$) in the interval ab , then in this interval (Whittaker and Robinson 1940),

$$|f(x) - P_{n-1}(x)| \leq \frac{1}{n!} \left| \frac{d^n f}{dx^n} \right|_{a < x < b} \text{Max } |A_n(x)|, \quad \dots \quad (10)$$

where

$$A_n(x) = (x-x_1)(x-x_2) \dots (x-x_n), \quad \dots \quad (11)$$

and where it is assumed that $\frac{d^n f}{dx^n}$ is continuous.

To obtain the smallest upper limit for $|f(x) - P_{n-1}(x)|$ as given by (10), the points of interpolation $x_1, x_2, \dots, x_n$ must be such as to ensure the smallest value of $\text{Max}_{a < x < b} |A_n(x)|$.

The interval of interpolation ab may be assumed to correspond to $-1 \leq x \leq 1$, which can always be the case after a suitable change of variables. Then the smallest value of $\text{Max}_{-1 < x < 1} |A_n(x)|$ is obtained if the points $x_1, x_2, \dots, x_n$ are the zeros of the Tchebycheff polynomial of degree n , $T_n(x)$, or more precisely if

$$A_n(x) \equiv T_n(x).$$

This theorem is due to Tchebycheff (1899), and a short proof of it will be given here.

The Tchebycheff polynomial $T_n(x)$ is given by

$$\begin{aligned} T_n(x) &= \frac{\cos(n \cos^{-1} x)}{2^{n-1}} \\ &= x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_0^*. \end{aligned} \quad (12)$$

Its zeros are in the interval $(-1, +1)$, and are given by $x_j = \cos \frac{(2j-1)\pi}{2n}$, $j=1, 2, \dots, n$. Therefore $T_n(x)$ can be put in the form (11).

On the other hand, for $A_n(x)$ different from $T_n(x)$, we have

$$\text{Max}_{-1 \leq x \leq 1} |A_n(x)| > \text{Max}_{-1 \leq x \leq 1} |T_n(x)|. \quad (13)$$

This can be derived as follows.

From (12) it is seen that

$$\text{Max} |T_n(x)| = \frac{1}{2^{n-1}}$$

and

$$T_n\left(\cos \frac{m\pi}{n}\right) = \frac{(-1)^m}{2^{n-1}}, \quad m=0, 1, 2, \dots, n.$$

Let us now suppose that

$$\text{Max}_{-1 \leq x \leq 1} |A_n(x)| < \text{Max}_{-1 \leq x \leq 1} |T_n(x)|.$$

Then the polynomial $T_n(x) - A_n(x)$ will be positive at the points $x_m = \cos \frac{m\pi}{n}$ corresponding to m even, and negative at the points x_m

* The coefficients a_j are determined by putting

$$\begin{aligned} x &= \cos \theta \\ T_n(x) &= \frac{\cos n\theta}{2^{n-1}} = \frac{1}{2^n} [\exp(in\theta) + \exp(-in\theta)] \\ &= \frac{1}{2^n} [(\cos \theta + i \sin \theta)^n + (\cos \theta - i \sin \theta)^n] \\ &= \frac{1}{2^n} [(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n]. \end{aligned}$$

corresponding to m odd, and hence $T_n(x) - A_n(x)$ will change its sign n times. It must therefore have n zeros, which is impossible since $T_n(x) - A_n(x)$ is of $n-1$ th degree.

For a similar reason it is not possible to have

$$\text{Max}_{-1 \leq x \leq 1} |A_n(x)| = \text{Max}_{-1 \leq x \leq 1} |T_n(x)|.$$

Hence (13) has been proved, and the upper limit of $|f(x) - P_{n-1}(x)|$ as given by (10) is minimum when $A_n(x) \equiv T_n(x)$.

In this case

$$|f(x) - P_{n-1}(x)| \leq \frac{1}{n!} \frac{1}{2^{n-1}} \text{Max}_{-1 \leq x \leq 1} \left| \frac{d^n f}{dx^n} \right|,$$

and it is seen that the upper limit is much lower than when $P_{n-1}(x)$ is a finite Taylor series $\tau_{n-1}(x)$, in which case we have

$$|f(x) - \tau_{n-1}(x)| \leq \frac{1}{n!} \text{Max}_{-1 \leq x \leq 1} \left| \frac{d^n f}{dx^n} \right|.$$

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 Chapters I and II.

XX. *Properties of Slow Electrons in Polar Materials**.

By H. FRÖHLICH, H. PELZER† and S. ZIENAU,
Department of Theoretical Physics, The University, Liverpool‡.

[Received November 25, 1949.]

SUMMARY.

Using a variational method we have investigated the properties of the lowest energy levels in a range $\hbar\omega$ above the ground level of the system consisting of an electron and a continuous dielectric medium. The latter is supposed to have a single vibrational frequency $\omega/2\pi$ for long longitudinal polarization waves. The interaction between the electron and the medium then depends on three parameters, the static dielectric constant ϵ , the optical refractive index $\epsilon_\infty^{\frac{1}{2}}$, and ω . The replacement of a crystalline lattice by a continuum is a good approximation if the length $b = (\hbar/2m\omega)^{\frac{1}{2}}$ is large compared with the lattice distance, a condition which is usually fulfilled.

We find that the energy of the ground level may be considerably (compared with $\hbar\omega$) below the energy which the system would have in the absence of interaction. The electron can be found with equal probability at any point in the medium. The average polarization of the medium therefore vanishes at any point. This does not hold, however, for the average polarization at a given distance from the electron. This quantity varies at large distances as the polarization of a point charge, but shows deviations below distances of the order b . The energy of interaction depends very little on the average velocity of the electron. Slow electrons, therefore, behave very similarly to free electrons (§4). It follows then that self trapping in the lattice—a suggestion which has often been discussed—does not exist (§5).

Modified forms of previous formulæ of the mean free path of electrons are given in §6. It is shown, however, that the validity of the whole method used at present to calculate mean free paths requires further investigation.

§1. INTRODUCTION.

THROUGH combined theoretical and experimental work during the past twenty years it has been established that slow electrons in periodic crystal lattices behave in many respects like free electrons; the main influence of the periodic field is to replace the electronic mass by a slightly

* Based on Report L/T 221 of the British Electrical and Allied Industries Research Association (E.R.A.).

† On the research staff of the British Electrical and Allied Industries Research Association.

‡ Communicated by the Authors.

different effective mass. The effect of deviations from the periodicity, *e. g.* by thermal motion or through lattice defects can then be described as leading to scattering of otherwise free electrons. It might seem that ionic crystals differ in this respect from others. For in these crystals the motion of ions has a resonance at frequencies which makes their displacement by moving charges strongly dependent on the velocity of the latter. In fact it has been suggested that in view of their polarization an electron might be trapped near any point in the lattice; and that a certain energy would have to be supplied to transfer it into a state in which it can move more freely. Our calculations show that this is not the case.

Electronic motion in ionic crystals is of importance for a considerable number of physical processes. In the present paper, therefore, we have made a systematic study of the properties of the lowest energy levels of an electron, assuming it to be free except for reaction forces due to the enforced displacements of ions.

To obtain a qualitative idea of the interaction between an electron and the polarization produced by it let us first consider the behaviour of a moving point charge e [*cf.* Fröhlich and Pelzer (1950)] and then discuss the modifications required if the point charge is replaced by an electron. The polarization $\mathbf{P}$ in an ionic crystal can be considered as super-position of two components [*cf. e. g.* Fröhlich (1949)]: (i) the optical polarization $\mathbf{P}_0$ which is due to the displacement of bound electrons of the lattice with a resonance frequency in the optical or ultra-violet region; (ii) the infra-red polarization $\mathbf{P}_{ir}$ whose resonance frequency $\omega/2\pi$ lies in the infra-red and is connected with the displacement of ions. For the slow velocities v of the point charge in which we are interested the optical polarization is always excited to its static value whatever the velocity v . The interaction of $\mathbf{P}_0$ with the point charge (or the electron) therefore is of no interest to us as it does not influence the energy difference between states of different velocity. The infra-red polarization behaves differently in this respect because of the lower resonance frequency of ionic oscillations. At large distances from the point charge (position $\mathbf{r}_0$) the polarization $\mathbf{P}_{ir}$ is of course the same as in the static case. It is thus obtained by subtracting the optical polarization $\mathbf{P}_0$ from the total polarization $\mathbf{P}$, and can be derived from an electric potential $\Phi(\mathbf{r})$ by the equation

$$4\pi\mathbf{P}_{ir}(\mathbf{r}) = \text{grad } \Phi(\mathbf{r}), \quad \dots \dots \dots (1.1)$$

where according to electrostatics

$$\Phi(\mathbf{r}) = - \left(\frac{1}{\epsilon_\infty} - \frac{1}{\epsilon} \right) \frac{e}{|\mathbf{r} - \mathbf{r}_0|}, \quad \dots \dots \dots (1.2)$$

Here $\mathbf{r}$ is any point at sufficient distance from $\mathbf{r}_0$, ϵ is the static dielectric constant and ϵ_∞ is the dielectric constant at high frequencies where the ionic motion cannot follow. $\epsilon_\infty^{\frac{1}{2}}$ is the optical refractive index. An ion is displaced as in the static case only if its distance from the approaching

point charge is sufficiently larger than v/ω . The reason is that it takes a time of the order $1/\omega$ to displace an ion. Below this distance the point charge passes so quickly that its action on the ion can be considered as a shock, exciting oscillation after it has passed. In this case the potential at the position of the point charge is obtained by replacing $|\mathbf{r}-\mathbf{r}_0|$ in (1.2) by v/ω ; multiplication by e leads to the energy of interaction. To obtain the net interaction energy we must add the energy required to polarize the lattice. For harmonically bound ions this reduces the interaction energy to about half of the above value. Its order of magnitude is thus

$$-\frac{1}{2}\left(\frac{1}{e_\infty}-\frac{1}{\epsilon}\right)\frac{e^2\omega}{v} \quad \dots \quad (1.3)$$

If the point charge is replaced by an electron the uncertainty relation requires that we cannot confine it to a small region in space without a considerable uncertainty in its velocity. Let this region be of linear dimension a around a point $\mathbf{r}_0$. Thus at a large distance from $\mathbf{r}_0$ the potential is still given by expression (1.2). At short distances there are now two competitive effects which prevent the potential from falling below a certain value. Considering first the uncertainty in position only, the magnitude of the lowest value of the potential would be obtained by replacing $|\mathbf{r}-\mathbf{r}_0|$ in (1.2) by a . On the other hand, considering the effect of the velocity only, $|\mathbf{r}-\mathbf{r}_0|$ must be replaced by v/ω . We should expect that it is the mechanism which leads to the larger critical distance which decides the order of magnitude of the potential inside the region to which the electron is confined. The two distances are not independent, however, because according to the uncertainty relation $\Delta v \simeq \hbar/ma$ and Δv decides the order of magnitude of v if the point $\mathbf{r}_0$ is either at rest, or moving only slowly. A minimum for the critical distance exists, therefore, being the lower value of the two quantities v/ω and $\hbar/mv$. It is hence of the order

$$b = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}, \quad \dots \quad (1.4)$$

a length which will be of importance later on*. It now follows that a minimum value for the negative interaction energy should exist whose magnitude is of the order

$$W = \frac{1}{2}\left(\frac{1}{e_\infty}-\frac{1}{\epsilon}\right)\frac{e^2}{b} \quad \dots \quad (1.5)$$

It is suggestive to imagine the electron to be trapped by the potential field due to the polarization induced by itself, as was first done by Landau (1933) and later on followed up by others. The electron cloud is supposed to be centred around a fixed point in the medium; the exact distribution depends on the interaction with the polarization of the medium which in turn depends on the distribution of charge density. By making

* The exact numerical value has been chosen for later convenience.

polarization and density distribution self-consistent a solution is obtained which is supposed to have the following properties: (i.) it constitutes the lowest energy level of the system; (ii.) its mobility is so low and its effective mass so high that the electron can be considered as trapped; (iii.) by supplying a minimum energy the electron can be liberated into an unbound state, similar to the case that an electron is bound (trapped) by a permanent potential. These properties, if true, would be of great importance for the behaviour of electrons in ionic crystals. No attempts have been made so far to prove any of the three contentions. They are all objectionable, and we shall find in fact that none are valid (§5).

§ 2. CALCULATIONS; GENERAL METHOD.

Following the treatment in a previous paper [Fröhlich, (1937) quoted as (I.)] we shall investigate properties of a system consisting of an electron and a continuous dielectric medium possessing a single infra-red vibrational frequency. The electron is treated as free apart from its interaction with this medium. Any effect of a periodic lattice field might be taken into account by the use of an effective electronic mass. In the previous paper the scattering of a relatively fast (several eV.) electron was considered by treating its interaction with the medium as a small perturbation. At present we shall be mainly interested in a calculation of the position and properties of the ground level of the system, and of the subsequent levels in a range of the order of 1/10 eV. For this purpose we require the Hamiltonian H consisting essentially of three parts: (i) the energy of the dielectric, described by its polarization; (ii) the electron's energy of interaction with the polarization; (iii) the kinetic energy of the electron. As indicated in the introduction the energy of interaction between the electron and the optical polarization is the same for all states in which we are interested at present. It thus leads to a constant term in the Hamiltonian which we shall omit. The state of the dielectric is then entirely described by the space and time dependence of the infra-red polarization $\mathbf{P}_{ir}$. It is convenient to develop $\mathbf{P}_{ir}$ into a Fourier series of polarization waves using a term introduced on a previous occasion (I.). Thus we can write

$$\mathbf{P}_{ir} = \frac{1}{V^{\frac{1}{2}}} \sum_{\mathbf{w}} (\mathbf{a}_{\mathbf{w}} \exp \{i\mathbf{w}\mathbf{r}\} + \text{c.c.}), \quad . \quad . \quad . \quad . \quad (2.1)$$

where V denotes the volume in which we are interested and c.c. stands for the complex conjugate of the preceding expression. In the absence of external forces (*e.g.* due to the electron) the $\mathbf{a}_{\mathbf{w}}$ depend periodically on time. As in (I.) longitudinal waves only are of interest because transverse waves do not interact with an electron. In contrast to (I.) we shall be mainly interested in polarization waves which are long compared with the lattice distance. It is for this reason that the dielectric can be treated as a continuum. The frequency $\omega/2\pi$ of all waves can then be considered as independent of wavelength. This frequency differs however from that of very short waves; its value will be discussed in §4.

It is useful to introduce the electric potential $\Phi(\mathbf{r})$ from which the polarization $\mathbf{P}_{ir}$ can be derived according to (1.1) because then the energy of interaction is simply given by

$$H_{\text{int}} = e\Phi(\mathbf{r}_e), \quad (2.2)$$

where $\mathbf{r}_e$ is the position of the electron. Introducing two new sets of coordinates X_w, Y_w and a constant M by

$$X_{\mathbf{w}} = a_{\mathbf{w}} + a_{\mathbf{w}}^*, \quad Y_{\mathbf{w}}/M\omega = -i(a_{\mathbf{w}} - a_{\mathbf{w}}^*), \quad . \quad . \quad . \quad (2.3)$$

we obtain from (1.1) and (2.1)

$$\Phi(\mathbf{r}) = \frac{4\pi}{V^{\frac{1}{3}}} \sum_{\mathbf{w}} \frac{1}{w} \left(X_{\mathbf{w}} \sin \mathbf{w}\mathbf{r} + \frac{Y_{\mathbf{w}}}{M_{\omega}} \cos \mathbf{w}\mathbf{r} \right). \quad (2.4)$$

Clearly without a field $Y_{\mathbf{w}}=M\dot{X}_{\mathbf{w}}$. The constant M was introduced so that formally the medium is represented by a set of harmonic oscillators with mass M , displacement $X_{\mathbf{w}}$ and momentum $Y_{\mathbf{w}}$. We shall choose the constant M in such a way that the oscillator energy is given by

$$H_{\text{osc}} = \sum_{\mathbf{w}} \frac{1}{2} \left(M \omega^2 X_{\mathbf{w}}^2 + \frac{Y_{\mathbf{w}}^2}{M} \right). \quad (2.5)$$

It should be clear, however, that neither has M the dimension of a mass nor X and Y those of length and momentum respectively.

Denoting by $\mathbf{p}$ the momentum of the electron the total Hamiltonian is given by

$$H=H_{\text{osc}}+e\Phi(\mathbf{r}_e)+\frac{p^2}{2m}. \quad (2.6)$$

The $X_{\mathbf{w}}$ and $Y_{\mathbf{w}}$ and $\mathbf{r}_e$ and $\mathbf{p}$ can then be treated as sets of canonical conjugate coordinates.

We might determine M from the known energy of a simple (*e.g.* homogeneous) displacement. We shall instead calculate the potential due to a point charge at $\mathbf{r}_0$ and determine M by comparison with the known result (1.2). This will give us an opportunity of demonstrating the method of handling (2.6) in a simple case. The replacement of the electron by a point charge implies that the kinetic energy term $p^2/2m$ must be removed from (2.6) and $\mathbf{r}_e$ must be replaced by the constant $\mathbf{r}_0$. Then if we introduce the constants

$$\bar{X}_{\mathbf{w}} = -\frac{4\pi e}{V^{\frac{1}{2}}} \frac{1}{w} \frac{\sin \mathbf{w} \mathbf{r}_0}{M \omega^2}, \quad \bar{Y}_{\mathbf{w}} = -\frac{4\pi e}{V^{\frac{1}{2}}} \frac{1}{w} \frac{\cos \mathbf{w} \mathbf{r}_0}{\omega}, \quad . \quad . \quad (2.7)$$

clearly

$$H = \sum_{\mathbf{w}} \left(H_{\mathbf{w}} - \frac{1}{2} M \omega^2 \bar{X}_{\mathbf{w}}^2 - \frac{1}{2} \frac{\bar{Y}_{\mathbf{w}}^2}{M} \right), \quad (2.8)$$

where

$$H_{\mathbf{w}=\frac{1}{2}} M \omega^2 (X_{\mathbf{w}} - \bar{X}_{\mathbf{w}})^2 + \frac{1}{2M} (Y_{\mathbf{w}} - \bar{Y}_{\mathbf{w}})^2. \quad (2.9)$$

By this transformation our Hamiltonian has been separated into a constant term and a set of Hamiltonians of harmonic oscillators with coordinates $X_{\mathbf{w}} - \bar{X}_{\mathbf{w}}$ and momenta $Y_{\mathbf{w}} - \bar{Y}_{\mathbf{w}}$.

Quantum-mechanically, $\bar{X}$ and $\bar{Y}$ are the expectation values of X and Y ; classically they are the averages over one period. The expectation value for $\Phi(\mathbf{r})$, using equation (2.4) is thus given by

$$\Phi(\mathbf{r} - \mathbf{r}_0) = -\frac{16\pi^2 e}{VM\omega^2} \sum_{\mathbf{w}} \frac{\cos(\mathbf{w} \cdot \mathbf{r} - \mathbf{r}_0)}{\mathbf{w}^2}. \quad (2.10)$$

On this, and on all later occasions, the sum will be transformed into an integral in the usual way by counting the number of values which $\mathbf{w}$ can have in a small range $w^2 dw d\Omega$ on the assumption of a periodic boundary condition. Thus

$$\sum_{\mathbf{w}} (\quad) \rightarrow \frac{V}{(2\pi)^3} \int w^2 dw d\Omega (\quad), \quad (2.11)$$

where $d\Omega$ is the element of the solid angle in $\mathbf{w}$ -space. The integration goes over all $\mathbf{w}$ -space*. The result agrees with (1.2) if we determine M from

$$\frac{1}{M\omega^2} = \frac{1}{4\pi} \left(\frac{1}{\epsilon_{\infty}} - \frac{1}{\epsilon} \right). \quad (2.12)$$

Unfortunately when replacing the point charge by an electron this simple method can no longer be used and a suitable approximative method has to be found. The electron as well as the oscillators will be treated quantum-mechanically.

Let us first express the Hamiltonian in a useful matrix representation. We shall make use of the complete orthogonal set of wave functions $\Psi_0(\mathbf{K}; n_1, n_1, \dots, n_{\mathbf{w}}, \dots)$ which diagonalizes H when the interaction term $e\Phi(\mathbf{r}_e)$ in (2.6) is neglected. These functions have previously been discussed in (I.). They can be written as

$$\Psi_0(\mathbf{K}; n_1, \dots, n_{\mathbf{w}}, \dots) = \frac{\exp\{i\mathbf{K}\mathbf{r}\}}{V^{\frac{1}{2}}} \prod_{\mathbf{w}} \chi_{\mathbf{w}}(X_{\mathbf{w}}, n_{\mathbf{w}}), \quad (2.13)$$

where $\chi_{\mathbf{w}}(n_{\mathbf{w}})$ is the wave function of a harmonic oscillator with coordinate $X_{\mathbf{w}}$ in its $n_{\mathbf{w}}$ th excited state. In this representation, the diagonal elements of the interaction, H_{int} , vanish. Thus the diagonal elements of H are given by

$$E_0(\mathbf{K}; n_1, \dots) = \frac{\hbar^2 \mathbf{K}^2}{2m} + \hbar\omega \sum_{\mathbf{w}} (n_{\mathbf{w}} + \frac{1}{2}), \quad (2.14)$$

corresponding to the energy when interaction is neglected. The non-diagonal elements vanish except for those corresponding to the absorption or emission of one quantum $\hbar\omega$ by the electron. If $\mathbf{w}$ is the wave number

* In (I.) it was necessary to introduce an upper limit w_0 of w . This is unnecessary at present as the main contributions to all integrals required will be found to arise from much smaller values of w .

of this quantum then they will be denoted by $(e\Phi)_{\mathbf{K}\mathbf{K}'\mathbf{w}}$ to show that only the interaction energy gives a contribution. Their value is found in a straight forward way similar as in (I., p. 236) by making use of (2.4) and of the well known matrix elements of the coordinate and the momentum of a harmonic oscillator. Introducing the value (2.12) for M we find

$$(e\Phi)_{\mathbf{K}\mathbf{K}'\mathbf{w}} = (e\Phi)_{\mathbf{w}} \begin{cases} (1+n_{\mathbf{w}})^{\frac{1}{2}} \\ -n_{\mathbf{w}}^{\frac{1}{2}} \end{cases}, \quad . \quad . \quad . \quad . \quad . \quad (2.15)$$

where

$$(e\Phi)_{\mathbf{w}} = \frac{4\pi ie}{V^{\frac{1}{2}}} \frac{1}{w} \left[\frac{\hbar\omega}{8\pi} \left(\frac{1}{\epsilon_{\infty}} - \frac{1}{\epsilon} \right) \right]^{\frac{1}{2}} . \quad . \quad . \quad . \quad . \quad (2.16)$$

Here $(1+n_{\mathbf{w}})^{\frac{1}{2}}$ holds in the case of an emission of a quantum $\hbar\omega$ with wave number $\mathbf{w}$ by the electron, provided

$$\mathbf{K}' = \mathbf{K} - \mathbf{w}. \quad . \quad . \quad . \quad . \quad . \quad (2.17)$$

For absorption $n_{\mathbf{w}}^{\frac{1}{2}}$ must be used instead, and

$$\mathbf{K}' = \mathbf{K} + \mathbf{w}. \quad . \quad . \quad . \quad . \quad . \quad (2.18)$$

All non-diagonal elements for which neither of these conditions is fulfilled vanish.

We shall derive our solutions from those of ordinary second order perturbation theory, which starts in zero order with an electron with energy smaller than $\hbar\omega$ and with all $n_{\mathbf{w}}=0$. Such an electron cannot emit a real quantum. It can, however, emit, and reabsorb virtually single quanta provided (2.17) and (2.18) are fulfilled respectively. The opposite sequence, first absorption and then emission is impossible as initially all $n_{\mathbf{w}}$ vanish. After previous emission, just one $n_{\mathbf{w}}=1$ is present however. Apart from the sign, both matrix elements are then equal, putting in (2.15) $n_{\mathbf{w}}=0$ and $n_{\mathbf{w}}=1$ respectively. They are then given by $\pm(e\Phi)_{\mathbf{w}}$, equation (2.16). The energies in zero order in the initial state $E_0(K)$ and in the intermediate state $E_0(\mathbf{K}, \mathbf{w})$ respectively are given by [making use of (2.14) and (2.17)]

$$E_0(K) = \frac{\hbar^2 K^2}{2m}, \quad E_0(\mathbf{K}, \mathbf{w}) = \frac{\hbar^2 (\mathbf{K} - \mathbf{w})^2}{2m} + \hbar\omega, \quad . \quad . \quad (2.19)$$

if we omit the zero point energy of the oscillators, $\Sigma \hbar\omega/2$. This is admissible as we are only concerned with energy differences. Thus, according to the well known formula, the perturbed energy is given by

$$\begin{aligned} E(K) &= E_0(K) - \sum_{\mathbf{w}} \frac{|(e\Phi)_{\mathbf{w}}|^2}{E_0(\mathbf{K}, \mathbf{w}) - E_0(K)} \\ &= \frac{\hbar^2 K^2}{2m} - \frac{2m}{\hbar^2} \sum_{\mathbf{w}} \frac{|(e\Phi)_{\mathbf{w}}|^2}{(\mathbf{K} - \mathbf{w})^2 + u^2 - K^2}, \quad . \quad . \quad . \quad (2.20) \end{aligned}$$

where we have introduced a constant u by (cf. 1.4)

$$u = \frac{1}{b}, \quad i. e. \quad \frac{\hbar^2 u^2}{2m} = \hbar\omega. \quad . \quad . \quad . \quad . \quad (2.21)$$

The corresponding wave function becomes

$$\Psi(\mathbf{K}) = \Psi_0(\mathbf{K}; 0, 0, \dots) + \sum_{\mathbf{w}} c_{\mathbf{w}} \Psi_0(\mathbf{K} - \mathbf{w}; 1_{\mathbf{w}}), \quad (2.22)$$

where

$$c_{\mathbf{w}} = - \frac{(e\Phi)_{\mathbf{w}}}{E_0(\mathbf{K}, \mathbf{w}) - E_0(K)} = - \frac{2m}{\hbar^2} \frac{(e\Phi)_{\mathbf{w}}}{(\mathbf{K} - \mathbf{w})^2 + u^2 - K^2}. \quad (2.23)$$

It consists of the sum of wave functions of our set having the same momentum $\hbar\mathbf{K}$ with appropriate factors. The first term corresponds to the case in which no quantum is excited, the remaining functions describe states with one excited quantum each.

We shall generalize the above expressions by continuing to use a wave function of the type (2.22) because without specializing the $c_{\mathbf{w}}$ this is the most general wave function with total momentum $\hbar\mathbf{K}$ containing functions with none or one excited quantum. It is to be hoped that this will suffice to describe the lowest energy levels, except in the cases of very strong interaction. The coefficients $c_{\mathbf{w}}$, and the energy $E(K)$ will be determined by minimizing the expectation value of the energy, with respect to $c_{\mathbf{w}}$. Thus

$$E(K) = \int \Psi^* H \Psi / \int \Psi^* \Psi. \quad (2.24)$$

The integration goes over all variables in Ψ . Making use of the matrix elements of H discussed above, we find after multiplication by $\int \Psi^* \Psi = 1 + \sum_{\mathbf{w}} |c_{\mathbf{w}}|^2$, (the latter holds in view of the orthogonality and normalization of the functions Ψ_0)

$$(1 + \sum_{\mathbf{w}} |c_{\mathbf{w}}|^2) E(K) = E_0(K) + \sum_{\mathbf{w}} |c_{\mathbf{w}}|^2 E_0(\mathbf{K}, \mathbf{w}) + \sum_{\mathbf{w}} (c_{\mathbf{w}} (e\Phi)_{\mathbf{w}} + \text{c.c.}). \quad (2.25)$$

Hence by varying with respect to $c_{\mathbf{w}}$ and $c_{\mathbf{w}}^*$ we obtain using (2.19) and (2.21)

$$c_{\mathbf{w}} = - \frac{(e\Phi)_{\mathbf{w}}}{E_0(\mathbf{K}, \mathbf{w}) - E(K)} = - \frac{(e\Phi)_{\mathbf{w}}}{\hbar^2[(\mathbf{K} - \mathbf{w})^2 + u^2]/2m - E(K)} \quad (2.26)$$

and

$$\begin{aligned} E(K) &= E_0(K) - \sum_{\mathbf{w}} \frac{|(e\Phi)_{\mathbf{w}}|^2}{E_0(\mathbf{K}, \mathbf{w}) - E(K)} \\ &= \frac{\hbar^2 K^2}{2m} - \frac{2m}{\hbar^2} \sum_{\mathbf{w}} \frac{|(e\Phi)_{\mathbf{w}}|^2}{(\mathbf{K} - \mathbf{w})^2 + u^2 - 2mE(K)/\hbar^2}. \end{aligned} \quad (2.27)$$

Both expressions (2.27) and (2.26) turn into the corresponding ones (2.22) and (2.20) in the perturbation theory if the energy perturbation is sufficiently small. We also notice that the normalization integral

$$J = \int \Psi^* \Psi = 1 + \sum_{\mathbf{w}} |c_{\mathbf{w}}|^2 \quad (2.28)$$

must differ very little from unity if perturbation theory holds. This is not required in the case of the variational method.

Finally we should like to mention that equation (2.27) for the lowest energy level with momentum $\hbar\mathbf{K}$ can also be obtained by diagonalizing the matrix representation of H with respect to states containing either none or one quantum.

§3. ENERGY, VELOCITY AND OTHER PROPERTIES.

The above expressions for the energy and the wave functions contain $\mathbf{K}$ as a parameter. The quantity $\hbar\mathbf{K}$ represents the momentum of the total system, electron+polarization waves, as can easily be verified. In fact each of the functions Ψ_0 appearing in (2.22) has the same momentum $\hbar\mathbf{K}$.

To obtain the energy in an explicit form it is useful to introduce the following dimensionless quantities [cf. (1.5), (1.4) and (2.21)]

$$x = \frac{K}{u}, \quad \alpha = \frac{W}{\hbar\omega} = \frac{1}{2} \left(\frac{1}{\epsilon_\infty} - \frac{1}{\epsilon} \right) \frac{e^2 u}{\hbar\omega}, \quad y = \frac{E_0(K) - E(K)}{\hbar\omega}. \quad (3.1)$$

The energy can then be written as

$$E(K) = \hbar\omega(x^2 - y) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.2)$$

if use is made of equation (2.19).

We then find from (2.27) by introducing from (2.16) and using (2.11), (2.21), (1.5) and (3.1),

$$\left. \begin{aligned} y &= \frac{\alpha u}{2\pi^2} \int_0^\infty dw \int \frac{d\Omega}{(\mathbf{K} - \mathbf{w})^2 + u^2(1+y) - K^2} \\ &= \frac{\alpha}{x} \sin^{-1} \frac{x}{(1+y)^{\frac{1}{2}}} \quad \text{if } 1+y > x^2 \\ &= \frac{\alpha}{x} \frac{\pi}{2} \quad \text{if } 0 < 1+y < x^2 \\ &= 0 \quad \text{if } 1+y < 0. \end{aligned} \right\} . \quad . \quad . \quad (3.3)$$

We shall restrict ourselves mainly to the case $1+y > x^2$ which as we shall see corresponds to the lowest range of energies. In this case the denominator in the above integral does not vanish for any real w . In the other cases the values given above are principal values. In a more convenient form we thus have

$$\frac{x}{(1+y)^{\frac{1}{2}}} = \sin \frac{yx}{\alpha} \quad \text{if } 1+y > x^2. \quad . \quad . \quad . \quad . \quad (3.4)$$

In particular, if y_0 denotes the value of y for $x=0$,

$$y_0(1+y_0)^{\frac{1}{2}} = \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.5)$$

From this expression we see that α acts as an interaction parameter. If $\alpha \ll 1$, then $y \simeq \alpha$, a result which is identical with that obtained by perturbation theory.

Near $x=0$, the quantity y increases quadratically with x . By development one obtains

$$y - y_0 \simeq \frac{y_0}{2 + 3y_0} \frac{x^2}{3} \quad \text{if } x^2 \ll 1. \quad . \quad . \quad . \quad . \quad . \quad (3.6)$$

The upper limit of this range is reached when $\sin(yx/\alpha)$ in (3.4) becomes equal to unity, i. e. at values x_1 and y_1 satisfying

$$\frac{x_1}{(1 + y_1)^{\frac{1}{2}}} = 1, \quad x_1 y_1 = \alpha \frac{\pi}{2}. \quad . \quad . \quad . \quad . \quad . \quad (3.7)$$

Hence

$$x_1^2 - y_1 = 1, \quad x_1(x_1^2 - 1) = \alpha \frac{\pi}{2}, \quad . \quad . \quad . \quad . \quad . \quad (3.8)$$

showing that $x_1 > 1$, and with (3.2), that the energy at this upper limit is equal to $\hbar\omega$. Also developing near x_1, y_1

$$(y - x^2) - (y_1 - x_1^2) \simeq (3x_1^2 - 1)^2 \frac{x_1^2}{\alpha^2} (x - x_1)^2 \quad \text{if } (x - x_1)^2 \ll x_1. \quad (3.9)$$

Turning now to the average velocity of the electron we shall prove that

$$v_x = \frac{1}{\hbar} \frac{\partial E(K)}{\partial K_x}, \quad . \quad . \quad . \quad . \quad . \quad (3.10)$$

where v_x represents the X -component. By definition,

$$v_x = \frac{\hbar}{mi} \int \Psi^* \frac{\partial \Psi}{\partial x} / \int \Psi^* \Psi, \quad . \quad . \quad . \quad . \quad . \quad (3.11)$$

or introducing from (2.22) using the orthogonality of the functions Ψ_0 ,

$$\begin{aligned} \frac{\hbar}{m} v_x &= \left(K_x + \sum_{\mathbf{w}} (K_x - w_x) |c_{\mathbf{w}}|^2 \right) / \left(1 + \sum_{\mathbf{w}} |c_{\mathbf{w}}|^2 \right) \\ &= K_x - \sum_{\mathbf{w}} w_x |c_{\mathbf{w}}|^2 / \left(1 + \sum_{\mathbf{w}} |c_{\mathbf{w}}|^2 \right). \quad . \quad . \quad . \quad . \quad . \quad (3.12) \end{aligned}$$

Now

$$w_x = w(\cos \theta \cos \vartheta - \sin \theta \sin \vartheta \cos \phi),$$

if θ is the angle between $\mathbf{w}$ and $\mathbf{K}$, ϑ the angle between $\mathbf{K}$ and the x -axis,

$$K_x = K \cos \vartheta, \quad . \quad . \quad . \quad . \quad . \quad (3.13)$$

and ϕ is an azimuth around the K -axis. Since $c_{\mathbf{w}}$ is independent of this azimuth ϕ we can average in (3.12) over ϕ , thus transforming $\mathbf{w}_x$ into $w \cos \theta \cos \vartheta$. With the use of (3.13) we then find

$$\frac{m}{\hbar} v_x - K_x = - \frac{K_x}{K} \sum_{\mathbf{w}} w |c_{\mathbf{w}}|^2 \cos \theta / \left(1 + \sum_{\mathbf{w}} |c_{\mathbf{w}}|^2 \right). \quad . \quad . \quad (3.14)$$

On the other hand, from (2.27) we find by using (2.26),

$$\begin{aligned} \frac{m}{\hbar^2} \frac{\partial}{\partial K_x} (E(K) - E_0(K)) &= \frac{m}{\hbar^2} \frac{\partial E(K)}{\partial K_x} - K_x = \frac{m}{\hbar^2} \frac{K_x}{K} \frac{\partial}{\partial K} \sum_{\mathbf{w}} \frac{|(e\Phi)_{\mathbf{w}}|^2}{E_0(\mathbf{K}, \mathbf{w}) - E(K)} \\ &= \frac{K_x}{K} \sum_{\mathbf{w}} |c_{\mathbf{w}}|^2 \left(K - w \cos \theta - \frac{m}{\hbar^2} \frac{\partial E(K)}{\partial K} \right). \quad (3.15) \end{aligned}$$

Hence the expression $(m/\hbar^2)(\partial E/\partial K_x) - K_x$ becomes equal to the right hand side of (3.14); equation (3.10) then follows immediately.

We shall now calculate the value of the normalization integral J (2.28) which is required to judge the accuracy of the perturbation method. By inserting from (2.26) with the use of (2.16), (2.11), (2.21) and (3.1) we can write

$$J = 1 + \sum_{\mathbf{w}} |c_{\mathbf{w}}|^2 = 1 + \frac{\alpha}{4\pi^2} u^3 \int \frac{dw d\Omega}{[(\mathbf{K} - \mathbf{w})^2 - K^2 + u^2(1+y)]^2}. \quad (3.16)$$

A straight forward integration leads to

$$J = 1 + \frac{\alpha}{2(1+y)(1+y-x^2)^{\frac{1}{2}}}. \quad (3.17)$$

It will be noticed that at the lowest level $x=0$, $y=y_0$, the value of J becomes, using (3.5),

$$J_0 = 1 + \frac{\alpha}{2(1+y_0)^{\frac{1}{2}}} = \frac{2+3y_0}{2(1+y_0)}. \quad (3.18)$$

Thus near $x=0$, the perturbation method converges if the interaction parameter α is sufficiently small compared with unity. As we approach the upper limit of our energy range, however, J tends towards infinity in view of (3.8). The perturbation method thus becomes increasingly unsatisfactory when the variational method should still be reliable. At the upper limit of our range $x=x_1$, however, J tends to infinity which means that here the variational method also breaks down.

A further quantity of interest is the expectation value of the polarization potential (2.4) at a certain distance from the electron. We have considered this for the lowest level, $K=0$, only. For this purpose we first require the quantity

$$\Phi(\mathbf{r}, \mathbf{r}_e) = \int \Psi^* \Phi(\mathbf{r}) \Psi / \int \Psi^* \Psi, \quad (3.19)$$

where the integrals extend over all the coordinates except those of the electron. Thus with (2.4)

$$\Phi(\mathbf{r}, \mathbf{r}_e) = \frac{4\pi}{V^{\frac{1}{2}}} \sum_{\mathbf{w}} \frac{1}{w} \left(X_{\mathbf{w}}(\mathbf{r}_e) \sin \mathbf{w} \mathbf{r} + \frac{Y_{\mathbf{w}}(\mathbf{r}_e)}{M\omega} \cos \mathbf{w} \mathbf{r} \right), \quad (3.20)$$

where we define

$$X_{\mathbf{w}}(\mathbf{r}_e) = \int \Psi^* X_{\mathbf{w}} \Psi / \int \Psi^* \Psi, \quad (3.21)$$

and similarly for Y . Use of (2.22), (2.13) (replacing there $\mathbf{r}$ by $\mathbf{r}_e$) and (2.28) leads to

$$(1 + \sum_{\mathbf{w}} |c_{\mathbf{w}}|^2) X_{\mathbf{w}}(\mathbf{r}_e) = (c_{\mathbf{w}} \exp \{-i\mathbf{w}\mathbf{r}_e\} + c_{\mathbf{w}}^* \exp \{i\mathbf{w}\mathbf{r}_e\}) \left(\frac{\hbar}{2M\omega} \right)^{\frac{1}{2}}. \quad (3.22)$$

Here the square root is the ordinary oscillator matrix element for X , and the value (2.12) of M must be introduced. A similar expression is obtained for $Y_{\mathbf{w}}(\mathbf{r}_e)$. By inserting $c_{\mathbf{w}}$ from (2.26) the summation of (3.20) can be carried out with the help of (2.11). Making further use of (3.5) and (3.18) the result is

$$\Phi(\mathbf{r}, \mathbf{r}_e) = - \left(\frac{1}{\epsilon_{\infty}} - \frac{1}{\epsilon} \right) \frac{e}{|\mathbf{r} - \mathbf{r}_e|} (1 - \exp \{-|\mathbf{r} - \mathbf{r}_e| u(1 + y_0)^{\frac{1}{2}}\}) \frac{2}{2 + 3y_0}. \quad (3.23)$$

We notice that the energy of interaction is given by

$$e\Phi(\mathbf{r}_e, \mathbf{r}_e) = - \left(\frac{1}{\epsilon_{\infty}} - \frac{1}{\epsilon} \right) e^2 u \frac{2(1 + y_0)^{\frac{1}{2}}}{2 + 3y_0} = - \frac{4(1 + y_0)^{\frac{1}{2}}}{2 + 3y_0} W, \quad (3.24)$$

which for small values of y_0 becomes equal to $-2W$. A fraction of this energy has to be spent to polarize the lattice.

RESULTS AND DISCUSSION.

§ 4. PROPERTIES OF THE LOWEST ENERGY LEVELS.

The results which we have obtained in the previous calculations concern the lowest energy levels of the system consisting of one electron and of a continuous dielectric medium possessing a proper frequency $\omega/2\pi$. Apart from the electronic charge e and mass m (which might be replaced by an effective mass to account for the periodic lattice field) they contain three parameters all of which can be found experimentally. These are the static dielectric constant ϵ , with optical refractive index $\epsilon_{\infty}^{\frac{1}{2}}$ and the frequency $\omega/2\pi$ of long longitudinal polarization waves. The latter can be expressed by the frequency $\omega_l/2\pi$ of long transverse waves which is the experimentally known frequency of infra-red residual waves. One finds (Fröhlich 1949, § 18)

$$\omega = \omega_l (\epsilon / \epsilon_{\infty})^{\frac{1}{2}}. \quad (4.1)$$

Long wave in this connection means long compared with the lattice distance. The calculations have shown that the length b introduced for dimensional reasons in (1.4) is the characteristic length occurring in our results. The fact that its ratio to the lattice distance

$$\frac{b}{a} = \frac{1}{a} \left(\frac{\hbar}{2m\omega} \right)^{\frac{1}{2}} \gg 1 \quad (4.2)$$

is large compared with unity for any reasonable value of ω justifies our procedure to treat the material as continuous.

The method which we have adopted contained a development of the polarization of the dielectric in normal coordinates. In the absence of electrons the behaviour of a dielectric material is then described in terms of quantized polarization waves. All quanta have the same energy $\hbar\omega$, but their momenta may differ, depending on the wavelength. The interaction with an electron can then be described in terms of absorption and emission of quanta $\hbar\omega$ by the electron ; this is in fact the procedure previously used to calculate the scattering of electrons (*cf.* I.). At present we are however, mainly concerned with the energy levels of our system. Their broadening by scattering processes would lead us at once into difficulties. We have escaped this, in the present paper, by restricting ourselves to the lowest energy levels in a range $\hbar\omega$ above the ground level ; in this case a free quantum can never exist. It should be realized that energy levels always refer to the energy of the whole system. To assume that not a single free quantum is excited in an infinite dielectric can only mean that its temperature is zero. We shall find below, however, that actually the only restriction required is to assume that

$$kT \ll \hbar\omega. \quad (4.2a)$$

In the mathematical technique which we have adopted we chose solutions which did not allow the existence of free quanta ; for technical reasons they break down at an energy somewhat higher than $\hbar\omega$ above the ground level. We shall present them in this whole range, but it should always be kept in mind that they are reliable only if the energy difference ΔE from the ground level is smaller than $\hbar\omega$,

$$\Delta E < \hbar\omega. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.3)$$

Lowest Levels.—According to (3.5), using (3.1) and (2.19) the lowest energy level $E(0)$ is obtained from the equation

$$-E(0)[1-E(0)/\hbar\omega]^{\frac{1}{2}}=W, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.4),$$

i. e.

$$E(0) \cong -W \quad \text{if} \quad W \ll \hbar\omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.5),$$

and

$$E(0) \cong -(\hbar\omega)^{\frac{1}{3}} W^{\frac{2}{3}} \quad \text{if} \quad W \gg \hbar\omega. \quad (4.6)$$

The energy W depends on the three parameters ϵ , ϵ_∞ and ω according to equations (1.5) and (1.4). For practically interesting cases usually $\alpha = W/\hbar\omega$ is somewhat larger than unity (cf. §5). Also it should be remembered that $W \propto \omega^{\frac{1}{3}}$, i. e. $\alpha \propto 1/\omega^{\frac{1}{3}}$.

The lowest level is the lower limit of a continuum; the energy difference from the lowest level, $\Delta E(K)$, depends on the parameter K when $\mathbf{K}$ is the momentum of our system. This should be distinguished from the electronic contribution $m\mathbf{v}$ ($\mathbf{v}$ =electronic velocity) because the dielectric carries a part of the momentum. It follows from (3.1) and (2.19) that

$$\Delta E(K) = E(K) - E(0) = \frac{\hbar^2}{2m} K^2 + \hbar\omega(y_0 - y), \quad . \quad . \quad . \quad (4.7)$$

where y satisfies equation (3.4). In the neighbourhood of $K=0$, in particular, using (3.6) and (3.1),

$$\Delta E(K) \simeq \frac{\hbar^2}{2m} K^2 \left(1 - \frac{1}{3} \frac{y_0}{2+3y_0} \right). \quad (4.8)$$

The second term in the bracket takes account of the influence of the interaction between electron and dielectric. Since the quantity $y_0 = -E(0)/\hbar\omega$ is positive the value of the bracket is always between 1 and 8/9. Thus, though the shift of an energy level due to interaction may be considerable (compared with $\hbar\omega$) it depends only very little on the momentum.

It should be mentioned here that ordinary second order perturbation gives the value (4.5) for the lowest level, and replaces the above bracket by $(1-y_0/6)$. If this method were applied when the condition $y_0 < 1$ does not hold it would lead to the contradictory result that the state $K=0$ is not the lowest. This demonstrates the supremacy of the variational method.

Turning to larger values of K , the deviations of $\Delta E(K)$ from $\hbar^2 K^2/2m$ become increasingly larger. In fact the curvature of the curve representing $\Delta E(K)$ against K changes sign and ends with horizontal tangent at a value $K_1 = ux_1$ (cf. 3.1) of K which constitutes the limit beyond which the mathematical formalism breaks down. At this value, according to (3.8) and (3.1),

$$E(K_1) = \hbar\omega. \quad (4.9)$$

Since $E(0) < 0$ it follows that always

$$\Delta E(K_1) > \hbar\omega. \quad (4.10)$$

Hence (cf. 4.3) this upper region is outside the range of validity of our calculations.

Velocity.—Making use of (3.10) and (4.8), the average electronic velocity near the ground level is given by

$$v = \frac{\hbar K}{m} \left(1 - \frac{1}{3} \frac{y_0}{2+3y_0} \right). \quad (4.11)$$

For higher values it reaches a maximum and then falls to zero. Again these latter parts are outside the range of validity of our calculations.

Acceleration. An external field F accelerates our system. To discuss this question we imagine a wave packet to be formed so that we can approximately fix the position of the electron. Shifting it a distance Δx means a transfer of energy $eF\Delta x$ from the field to the electron. Hence, the value of K_x changes by an amount ΔK_x and the energy-treating ΔK_x as small—by $\Delta K_x \partial E(K)/\partial K_x$. Dividing by the time Δt required, we have with $v_x = \Delta x/\Delta t$ and with (3.10),

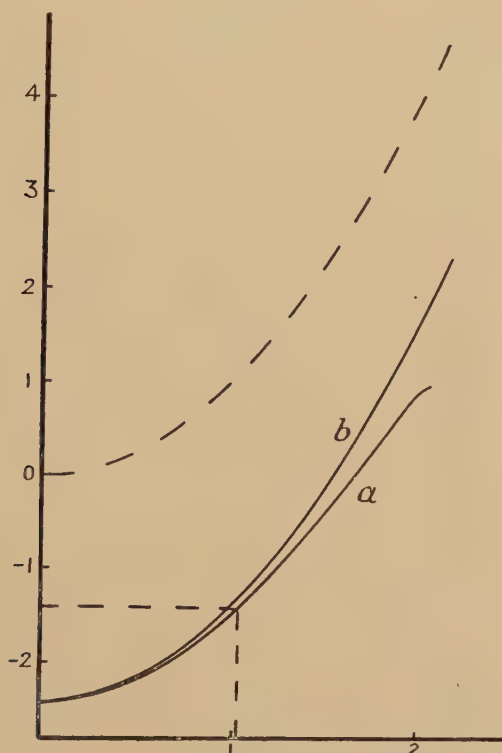
$$\frac{\Delta K_x}{\Delta t} = \frac{eF}{\hbar}, \quad (4.12)$$

an equation well known from the theory of metals. Using again (3.10) we find

$$\frac{\Delta v_x}{\Delta t} = \frac{1}{\hbar} \frac{\partial^2 E}{\partial K_x^2} \frac{\Delta K_x}{\Delta t} = \frac{eF}{\hbar^2} \frac{\partial^2 E}{\partial K_x^2}. \quad (4.13)$$

In the region in which (4.8) and (4.11) hold the acceleration is thus similar to that of a free electron. For larger values of K it decreases, reaches zero when the velocity has its maximum and then reverses its sign. The latter region again is outside the range of validity of our calculations.

Fig. 1.



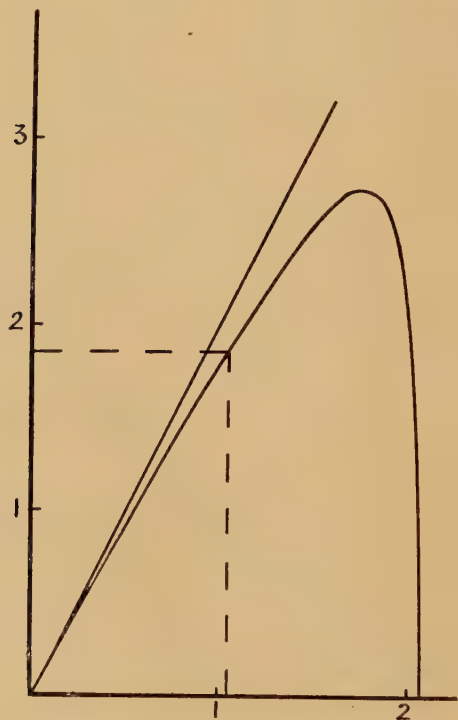
Curve a represents the energy of electrons in units of $\hbar\omega$ (i. e. the quantity $x^2 - y$) as a function of the reduced momentum x ; the dotted curve refers to free electrons. Curve b is the same curve shifted so as to demonstrate that the interaction hardly depends on the momentum. Validity outside the dotted region is doubtful.

The main results of our calculations are shown in the accompanying figures. The quantities $\hbar\omega$ and $\hbar u$, respectively, have been used as units of energy and momentum. The results according to §3 then depend on the interaction parameter only. The value $\alpha = 4.47$ has been used in the drawings (for NaCl it would be 5.2). Fig. 1 represents the energy as function of the momentum. For comparison a free electron curve

has been drawn showing a shift considerably larger than $\hbar\omega$ (*i. e.* unity in the present units). A displaced free electron curve starting at the same ground level demonstrates that the shift due to interaction is nearly independent of the momentum. The validity of the energy curve is restricted to values in a range $\hbar\omega$ above the ground level.

Fig. 2 shows the average electronic velocity as a function of momentum, compared with the case of free electrons. Within the range of validity the deviations are seen to be small.

Fig. 2.



The full line is the average electronic velocity as a function of momentum ; the straight line corresponds to free electrons. Validity outside dotted region is doubtful.

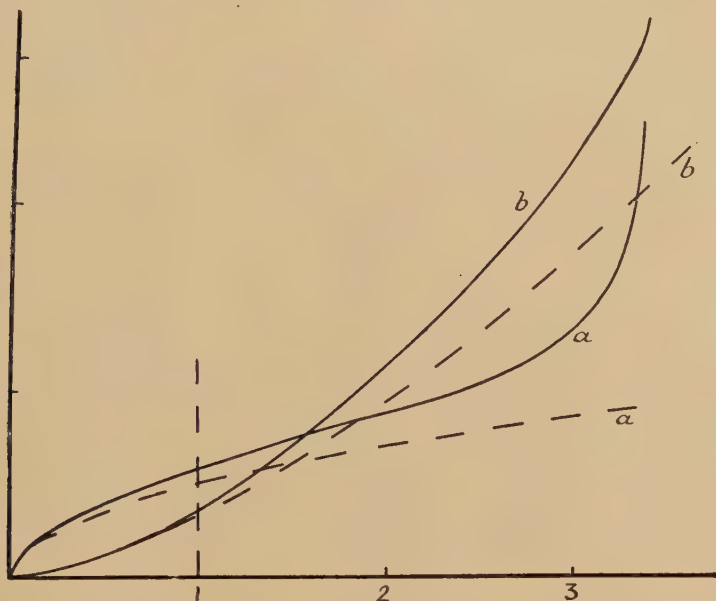
In fig. 3 we show the density of energy levels per unit energy $D(E)$ as a function of energy, and the total number of levels up to a given energy, $\int D(E) dE$. For comparison the corresponding curves in the case of free electrons are also drawn. The quantity $D(E)$ can easily be calculated from relation

$$D(E) \propto K^2 \frac{dK}{dE} (4.14)$$

§ 5. THE QUESTION OF SELF-TRAPPING.

As mentioned at the end of §1 it has first been suggested by Landau (1933) that an electron might "dig its own hole" and thus be trapped in the medium. The electron would thus be described by a probability distribution whose value is appreciable only in the neighbourhood of a given point in the medium. It will then give rise to a certain polarization which if frozen in, would act as a centre of attraction for the electron and lead to at least one bound state. Probability distribution and polarization can be made self consistent. The result is a particular solution of (2.6).

Fig. 3.



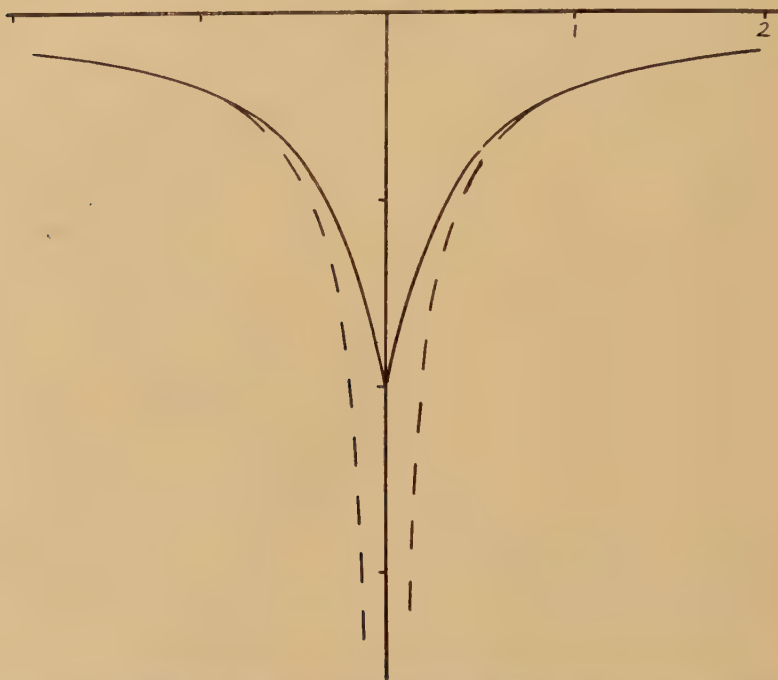
Curve *a* represents the density of energy levels as a function of energy ; *b* shows the total number of levels. The dotted curves refer to free electrons. Validity on the right of the vertical line is doubtful.

One of the objections to be raised against the above method is the preference given to a particular point in the medium. No doubt, in the exact solutions for stationary states the electron can be found with equal probability at any point. Our solution (2.22), (2.26) satisfies this requirement. At the same time the polarization relative to the position $\mathbf{r}_e$ of the electron (*cf.* fig. 4) shows a very similar behaviour as it does in the Landau method relative to a fixed point. The polarization potential $\Phi(\mathbf{r}, \mathbf{r}_e)$ behaves for large distances like $1/|\mathbf{r} - \mathbf{r}_e|$ down to distances of the order *b* (provided α is not very big). For smaller distances $\Phi(\mathbf{r}, \mathbf{r}_e)$ varies only slowly as compared with $1/|\mathbf{r} - \mathbf{r}_e|$.

A second objection arises when the linear extension of the probability distribution is larger than our characteristic distance b , equation (1.4). In this case it is no longer possible to assume the polarization to be rigid. This objection mainly holds for small values of the interaction parameter α [cf. (3.1)], or if α is large, for excited states whose orbit is larger than b .

Comparison of our results for the lowest energy level $E(0)$ [equation (4.4)] with those obtained by Pekar (1946) (using the Landau method) confirms the above conjecture in the case that the quantity $c=(1/\epsilon_\infty-1/\epsilon)^{-1}$, and hence (cf. 3.1) α , is small compared with unity. For in this case Pekar's

Fig. 4.



The full curve represents the average potential (3.23) as a function of a reduced distance $\xi=|\mathbf{r}-\mathbf{r}_c|u(1+y_0)^{\frac{1}{2}}$ from the electron; the dotted curve refers to the case of a point charge.

value for $|E(0)|$ is proportional to c^2 and therefore of a smaller order than ours which [cf. (4.5) and (1.5)] is proportional to the first power of c . Our value for $|E(0)|$ on the other hand, being based on a variational method, is certainly not too large.

For strong interaction (large α), both methods lead to similar values for $E(0)$. Thus Markham and Seitz (1948) using methods developed by Mott and Littleton (1938) and by Pekar (1946) find the value of -0.13 eV. as compared with -0.09 in our calculations*.

* Obtained from equation (4.4) with the help of (1.5) and (1.4) using the following numerical values: $\epsilon=5.6$, $\epsilon_\infty=2.3$, $\omega=4.8 \times 10^{13}$ sec $^{-1}$. The latter [cf. (4.1)] corresponds to a residual wavelength $(2\pi c_0/\omega)(\epsilon/\epsilon_\infty)^{\frac{1}{2}}=61\mu$, c_0 =velocity of light.

The case of the Landau method for a calculation of the lowest energy level thus can be expected to lead to reasonable values only if the interaction parameter α is large. This method as described so far does not yet lead to self-trapping. Two further assumptions would have to be made which have never been proved. They are both objectionable and our calculations prove in fact that they are wrong. They are

- (i.) that the whole configuration cannot move freely through the lattice,
- (ii.) that there would be a continuum above the (presumed) discrete ground level similar to the case of an electron moving in a permanent field of force.

This latter assumption would lead to an infinitely extended probability distribution for which, as we have shown above, the Landau method cannot be applied.

Assumption (i.) is disproved by our calculations which show that the energy difference of states with non-vanishing momentum $\hbar K$ from the ground state ($K=0$) hardly differs from that for free electrons although the shift of energy levels due to the interaction may be large. That assumption (i.) cannot be correct can also be seen more directly from the fact that the polarization due to a moving point charge differs from that of a static one only at distances below v/ω . For an extended charge distribution, representing the electron in the Landau method, no appreciable dependence of the polarization on the velocity can be expected so long as v/ω is smaller than the main extension of the charge. In this case, then, the interaction energy (including the energy required to polarize the medium) is nearly independent of velocity.

It should be added that our method appears to be quantitatively correct for values of α up to $\alpha \simeq 1$. For larger values ($\alpha \simeq 5$ for NaCl) we hope that the variational method still ensures a correct order of magnitude. Improved calculations are in progress to investigate this.

§ 6. MEAN FREE PATH.

In calculating the conductivity the average contribution per electron is usually written in the form $e^2\bar{\tau}/m$ where $\bar{\tau}$ is a quantity with the dimension of a time, the average relaxation time. The quantity $e\bar{\tau}/m$ is often denoted as mobility. If a model is used in which electrons are free to move except for occasional elastic collisions, then a relaxation time $\tau(v)$, depending on the velocity v can be introduced in such a way that its average over all electrons, $\overline{\tau(v)}$ is equal to $\bar{\tau}$,

$$\overline{\tau(v)} = \bar{\tau}. \quad (6.1)$$

Similarly a mean free path $l(v)$ and an average mean free path $\bar{l}$ can be introduced by

$$l(v) = \tau(v)v, \quad \bar{l} = \overline{l(v)}. \quad (6.2)$$

The relaxation time $\tau(v)$ is not simply the average time between two collisions but contains also the scattering angle; it might be called the average time between two large angle scatterings.

In our case of electrons in an ionic crystal the assumption that the interaction between electrons and polarization waves can be treated as a small perturbation leads to a straight forward calculation of the conductivity, and hence of the relaxation time although the collisions are not elastic and are connected with absorption and emission of quanta $\hbar\omega$. It turns out, however, that only for electronic energies large compared with $\hbar\omega$ (treated in I.) or smaller than $\hbar\omega$ (Fröhlich and Mott 1939) is this treatment reliable. Energies which are only slightly larger than $\hbar\omega$ lead to relaxation times which are smaller than $1/\omega$ or to mean free paths shorter than the de Broglie wavelengths as we shall see presently. This means that the method is then no longer valid.

To treat this case of energies of several $\hbar\omega$ the calculation in (I.) must be slightly modified. For in (I.) we were mainly concerned with fast electrons scattered primarily by very short polarization waves. Electrons with energies of only several $\hbar\omega$, however, absorb or emit only very long polarization waves as follows from conservation of energy and momentum. This means that the assumptions made at the beginning of this paper (especially §2) hold. Instead of carrying through the calculations in detail it is sufficient to compare with (I.) the expressions for the matrix elements and from there derive the replacements to be made in the final expression. The reason is that the Hamiltonian used in (I.) differs from ours only in the value of constant parameters. Comparing now the expression for $M_{kk'v}$ in (I. page 236) with our matrix element $(e\Phi)_{\mathbf{kk}'\mathbf{w}}$, [equations (2.15) and (2.16)] we see that we have to replace

$$(a^3N)^{\frac{1}{2}}, \quad \frac{e^2}{a^{\frac{3}{2}}}, \quad \frac{1}{(Mv)^{\frac{1}{2}}},$$

of (I.) by

$$V, \quad 2^{\frac{1}{2}}e, \quad \left[\frac{\omega}{2} \left(\frac{1}{\epsilon_{\infty}} - \frac{1}{\epsilon} \right) \right]^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (6.3)$$

respectively. Thus equations (15) and (16 a) of (I.) must be replaced by

$$\frac{1}{\tau} = \frac{1}{\tau_0} \left[1 + \frac{2}{\exp \{ \hbar\omega/kT \} - 1} \right] \quad . \quad . \quad . \quad . \quad . \quad (6.4)$$

and

$$\frac{1}{\tau_0} = \left(\frac{1}{\epsilon_{\infty}} - \frac{1}{\epsilon} \right) \frac{e^2\omega}{\hbar v}, \quad \text{if } E' > m \frac{v^2}{2} \gtrsim \text{several } \hbar\omega. \quad . \quad . \quad (6.5)$$

The energy E' (denoted by $2^{\frac{1}{2}}\pi^2\hbar^2/(8ma^2)$ in I.) is of the order of 1 eV. In thermal equilibrium the condition attached to (6.5) holds for an appreciable number of electrons only if $kT \gg \hbar\omega$. It then follows that

$$\frac{1}{\tau_0} = \left(\frac{1}{\epsilon_{\infty}} - \frac{1}{\epsilon} \right) \frac{2e^2kT}{\hbar^2v}, \quad \text{if } kT \gg \hbar\omega. \quad . \quad . \quad . \quad . \quad (6.6)$$

Hence using (6.2) and $\bar{v}^2 = 3kT/m$ we obtain

$$\bar{l} = \frac{3}{2} \frac{\epsilon\epsilon_\infty}{\epsilon - \epsilon_\infty} a_0 \quad \text{if } kT \gg \hbar\omega, \quad . \quad . \quad . \quad . \quad (6.7)$$

where

$$a_0 = \frac{\hbar^2}{me^2} \simeq 0.5 \times 10^{-8} \text{ cm.} \quad . \quad . \quad . \quad . \quad (6.8)$$

is the Bohr radius.

The remarkably simple result leads to values of $\bar{l}$ of the order of 5×10^{-8} cm. This is appreciably smaller than the de Broglie wavelength of thermal electrons at room temperature. The quantitative validity of (6.7) is, therefore, doubtful.

Electrons with energies below $\hbar\omega$ can only absorb but not emit quanta $\hbar\omega$. The perturbation calculation for this case has been carried through by Fröhlich and Mott (1939) modifying the method developed in (I.) so as to hold for very slow electrons. In the light of the present paper a further modification of the constant is required now, by making the replacement (6.3). This should be made in the second formula on page 501 of Fröhlich and Mott (1939) leading to replace

$$\frac{1}{\tau_0} = \frac{2}{3} \frac{e^4 k}{Ma^3 \hbar v k_v} \quad \text{by} \quad \frac{1}{\tau_0} = \frac{2}{3} \frac{\omega}{a_0 u} \left(\frac{1}{\epsilon_\infty} - \frac{1}{\epsilon} \right), \quad . \quad . \quad (6.9)$$

considering equation (6.8). Use has been made of the fact that k_v of the previous paper has now been denoted by u , cf. equation (2.21), and that $\hbar k/m = v$. Applying then equation (9) of Fröhlich and Mott and (2.21) we obtain for the mean free path

$$l = \tau v = \tau_0 v (\exp \{ \hbar\omega/kT \} - 1) = 3a_0 \frac{\epsilon\epsilon_\infty}{\epsilon - \epsilon_\infty} \left(\frac{mv^2}{2\hbar\omega} \right)^{\frac{1}{2}} (\exp \{ \hbar\omega/kT \} - 1). \quad (6.10)$$

The average mean free path thus becomes

$$\bar{l} = \frac{6}{\sqrt{\pi}} a_0 \frac{\epsilon\epsilon_\infty}{\epsilon - \epsilon_\infty} \left(\frac{kT}{\hbar\omega} \right)^{\frac{1}{2}} (\exp \{ \hbar\omega/kT \} - 1) \quad \text{if } kT \ll \hbar\omega. \quad . \quad (6.11)$$

It is larger than the one obtained previously [equation (11)] by a factor

$$\frac{\epsilon\epsilon_\infty}{\epsilon - \epsilon_\infty + 1}. \quad . \quad . \quad . \quad . \quad . \quad (6.12)$$

As a consequence of this correction there is a considerable improvement in the agreement between theoretical and experimental values of $\bar{l}$ of cuprous oxide. The previous formula gave values eight times too small; the factor (6.12) reduces this to about two ($\epsilon \simeq 10$). It is thus required to assume the effective mass to be only slightly smaller than the electron mass to obtain agreement.

The above formulæ for mean free paths have been derived on the assumption that perturbation theory is applicable. We found in §§ 2 and 3 that at the absolute zero of temperature this holds if the interaction parameter α given by equation (3.1) is small, $\alpha < 1$. This

result can easily be extended to higher temperatures as long as $kT \ll \hbar\omega$. For on this condition relatively few free quanta are excited; they influence only little the calculations of the energy of interaction so long as we deal with electrons whose kinetic energy, in zero approximation, is sufficiently far below $\hbar\omega$.

Thus although equation (6.11) seems well founded, if $\alpha < 1$, a serious objection can be raised. Absorption of a quantum $\hbar\omega$ leads the electron into a level from which after a very short interval it would re-emit a quantum $\hbar\omega$ —in general with different momentum. This suggests to treat such collisions as second order transitions, leading to the absorption of a quantum $\hbar\omega$, followed by a competitive emission of another quantum $\hbar\omega$ whose wave number $\mathbf{w}$ can have any value in a whole range. The formal method to be adopted would be very similar to that used by Breit and Wigner (1936) in nuclear physics*.

It should also be clear that by allowing the temperature T to be different from zero the energy of our whole system is always considerably higher than $\hbar\omega$ above the ground level. We have just seen that for low temperatures this would not seriously affect our calculations as long as perturbation theory is valid. This holds also for larger values of the interaction parameter α where the variational method differs considerably from the perturbation method, but only as far as the energy of interaction is concerned. Collisions would require an entirely new treatment. The former contention follows at once by considering in the variational calculations, especially in the sum in (2.27), only those wave numbers for which free oscillations are not excited thermally and then treating the latter by perturbation theory. This is permissible because their relative number is small so long as $kT \ll \hbar\omega$; the subsequent change in the expression for the energy [cf. (3.3)] is therefore small too. In the case of collision, however, the method cannot be applied because it is then no longer possible to separate the treatment of excited and nonexcited oscillators.

The authors are indebted to the British Electrical and Allied Industries Research Association for their support.

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* Added in proof.—One of us (S.Z.) has now carried out this calculation. It leads to a further factor $\frac{1}{2}$ in equation (6.11).

XXI. *Kinetics of the Phase Transition in Superconductors.*

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ABSTRACT.

A study is made of the influence of electromagnetic effects on the speed at which a transition can occur between the normal and superconducting states of a metal in a magnetic field, and it is concluded that these are powerful enough to be the dominant factor determining the speed. Illustrative examples include the transition of a superconducting plane slab and cylinder in a field greater than critical, the destruction of superconductivity in a wire by means of a current, and the mechanism whereby the intermediate state is established.

I. INTRODUCTION.

THE phase-transition of a superconductor into the normal state through the influence of a magnetic field has been extensively studied from the point of view of determining the conditions under which the two phases may exist side by side in equilibrium. Typical of the experimental work on this problem are determinations of critical magnetic fields; studies of the destruction of superconductivity in thin wires by passage of a current; and investigations of the intermediate state in bodies of non-zero demagnetizing coefficient. At the same time fundamental thermodynamical studies of the various problems have served to correlate the results of these experiments with other properties of the superconductor, such as the specific heat. Very much more attention has been directed to this, the statical aspect of the problem, than to the kinetic aspect, and scant consideration has been given to the mechanisms which are involved during the progress of the transition. Examples of the types of phenomena which demand kinetic treatment are such problems as the rate at which the transition from the superconducting to the normal state occurs in a field greater than the critical field, or from the normal to the superconducting state in a field less than critical; the closely allied problem of the rate of propagation of a phase boundary along a wire; and the problem of the mechanism whereby the intermediate state is established. Of the experimental work on this aspect of the phase transition we may cite as examples the investigation by Lazarev, Galkin and Khotkevich (1947) of the behaviour of wires carrying high-frequency alternating currents of such amplitude as to induce a phase transition, and the recent work of

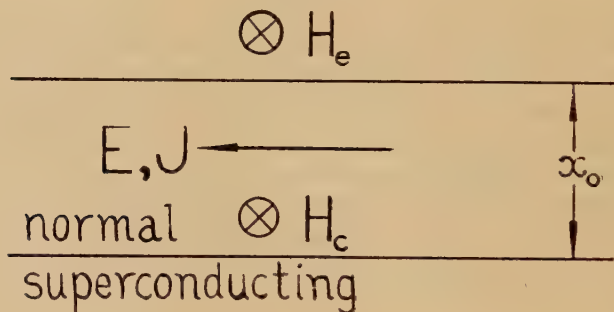
* Communicated by the Author.

Faber (1949) on the propagation of phase boundaries along wires. There appears to have been no discussion given of the electromagnetic processes which will have to be considered in any attempt to account for the observed phenomena, and the present paper represents an attempt to provide a starting-point for more rigorous treatments than can be given at the moment. It is unfortunate that the two experiments quoted are concerned with transitions which will be difficult to treat rigorously, and it is therefore not considered expedient to give any discussion of them here. On the other hand, the physical principles of the electromagnetic processes underlying these complicated examples may be understood clearly from idealized simple models, and the way may therefore be paved towards at least a qualitative understanding of the problems of more practical interest.

II. THE PHASE TRANSITION IN A PLANE SEMI-INFINITE SLAB.

Consider a semi-infinite slab of a superconducting material having a plane surface normal to the X-axis, as in fig. 1. If a magnetic field H_e ,

Fig. 1.



greater than the critical field H_c , be suddenly applied parallel to the surface, we may imagine that a thin layer at the surface makes a transition into the normal state, so that there is created a plane phase boundary parallel to the surface. The boundary will move into the interior of the metal as the normal region expands, and we wish to find its rate of propagation. It is clear that so long as the external field exceeds H_c there can never be a stable position of the boundary, since if it comes to rest the field at the boundary will eventually rise to a value H_c . On the other hand, the rate of propagation cannot be too great, since as the normal region expands magnetic flux must enter it, and in doing so induce currents whose effect is to reduce the field strength at the boundary. It seems reasonable to suppose, then, that the boundary moves at such a rate that the field strength next to it is maintained through the action of the induced currents at a value H_c .

For then a uniform current of density $\sigma H_c \frac{dx_0}{dt}$ will be induced by the movement of the boundary, so that in order to maintain a difference in field strength pH_c between the boundary and the surface it is necessary to put

$$4\pi\sigma H_c x_0 \frac{dx_0}{dt} = pH_c,$$

which leads immediately to equation (4). It will be found valuable to keep this approximate method in mind when we come to consider problems to which the above method of exact solution does not apply. The assumption that the field strength in the normal metal is everywhere H_c may be regarded as the first stage in the solution of the problem by successive approximation. We have seen that this assumption leads to a uniform current density in the normal metal, and we may now use this result to derive a closer approximation to the field distribution. The next stage in the calculation will thus start from the assumption that the field varies linearly from H_c at the phase boundary to $H_c(1+p)$ at the surface. A closer approximation to the current distribution may then be reached immediately, leading to the result of equation (3) with α put equal to $3/(3+p)$. This expression is shown as a dotted line in fig. 2, from which it is clear that the second approximation is close to the exact solution over a wide range of p .

It will be noted that the depth of the boundary, given by equation (3), varies as the square root of the time, as is to be expected from equation (1), which has the same form as the diffusion equation. Equations (3) and (4) bear a resemblance to the result obtained in the theory of the skin effect, that the depth of penetration of a field alternating with angular frequency ω is given by the expression :

$$\delta^2 = 1/2\pi\omega\sigma.$$

Thus, if we use the approximate solution (4), we see that for an external field $2H_c$ (*i. e.* $p=1$), the depth of the phase boundary after a time t is the same as the skin depth in the normal metal for a frequency $1/t$. For good conductors the rate of progress of the boundary is consequently very slow. For example, a specimen of pure tin may have in the normal state a conductivity of $5 \times 10^8 \text{ ohm}^{-1} \text{ cm.}^{-1}$, or in e.m.u., $\sigma=0.5$. If the external field exceeds the critical field by 10 per cent, $p=0.1$, and equation (3) becomes

$$x_0^2 \doteq 0.03t.$$

A time of about 30 seconds is therefore needed for the boundary to penetrate 1 cm., and nearly one hour to penetrate 10 cm.

In this treatment we have considered the transition to be isothermal, although, as is well known, the adiabatic transition from the superconducting to the normal phase results in a drop in temperature. It is

therefore necessary to consider whether it is justifiable to assume that if the outside surface of the material is maintained at constant temperature throughout, by contact with liquid helium, there is sufficiently good thermal conductivity to maintain isothermal conditions at all points. On account of the formal similarity between equation (1) and the equation of heat flow the analysis may be carried out by dimensional means. In order that the transition shall be isothermal, all that is necessary is that the skin depth for a heat wave at a given frequency shall be much greater than the corresponding skin depth for an electromagnetic field in the normal material. That is, $\left(\frac{4\pi\sigma\kappa}{C}\right)^{\frac{1}{2}} \gg 1$, where κ is the thermal conductivity and C the specific heat per unit volume. For a good specimen of tin at 3° K the quantity on the left-hand side takes a value of about 500, so that the assumption of isothermal conditions is well satisfied. Since however $\sqrt{\sigma\kappa}$ is proportional to the conductivity, it might be necessary to consider thermal effects in specimens of poor conductivity.

III. THE PHASE TRANSITION IN A CYLINDER.

Consider a long superconducting cylinder of radius a suddenly subjected to a longitudinal field $H_c(=H_c(1+p))$. Ideally we may suppose that a normal sheath will be formed at the surface, and the cylindrical superconducting core will shrink until finally it vanishes at the centre. The equation governing the rate of shrinkage will be the same as equation (1) but now on account of the cylindrical symmetry the method of solution adopted for the plane slab becomes inapplicable. We shall therefore adopt the approximate method mentioned in the last section to solve the problem, assuming the field strength in the normal material to take everywhere the value H_c .

Let the radius of the superconducting core be r_0 ; then the magnetic flux contained in the normal material between the core and a cylinder of radius r :

$$\phi = \pi(r^2 - r_0^2)H_c.$$

As the core contracts, the induced electric field at a radius r will be given by

$$E(r) = -\frac{1}{2\pi r} \frac{d\phi}{dt} = H_c \frac{r_0}{r} \frac{dr_0}{dt},$$

The circulating current density at radius r is thus given by

$$J(r) = \sigma H_c \frac{r_0}{r} \frac{dr_0}{dt}.$$

Now if the external field is $H_c(1+p)$ we must have that

$$4\pi \int_{r_0}^a J dr = -pH_c.$$

Hence

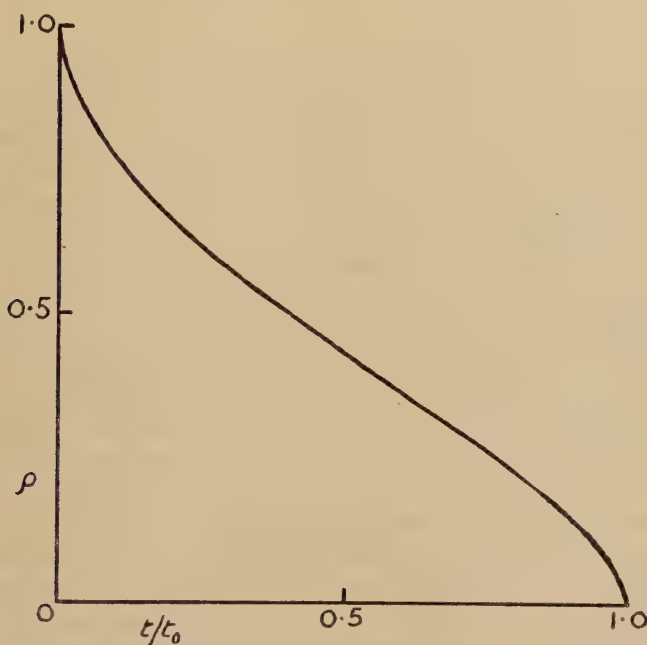
$$r_0 \log \frac{a}{r_0} \cdot \frac{dr_0}{dt} = -p/4\pi\sigma.$$

Integrating, and putting $r_0=a$ when $t=0$, we obtain the result :

$$1-\rho^2(1-2\log\rho)=\frac{t}{t_0}, \quad \text{where} \quad \rho=r_0/a \quad \text{and} \quad t_0=\pi\sigma a^2/p.$$

When $\rho=0$, $t=t_0$, so that t_0 represents the total time required for destruction of superconductivity in the cylinder. Comparing this result with equation (4), we see that it is just one-half of the time needed for the penetration of a plane boundary to a depth r_0 . The manner in which the superconducting core shrinks is illustrated in fig. 3, where ρ is plotted against t/t_0 . The initial stage of the shrinkage approximates closely to the behaviour of the plane boundary, but after the radius of the core has

Fig. 3.



fallen below one-half of the specimen radius the rate of shrinkage increases one more, until in the final stage the last traces of superconductivity vanish rapidly.

The reverse process, the transition from the normal to the superconducting phase in a field less than critical, is complicated by the occurrence of "supercooling", the metastable persistence of the normal phase in a field slightly less than critical. It is no part of this discussion to enquire into the mechanism responsible for supercooling, which can only be understood in terms of a detailed theory of the superconducting state, and of the surface energies at the boundaries between the two phases.

If, however, we suppose that eventually the supercooled phase breaks down by the formation of a small superconducting nucleus, its subsequent growth is probably largely determined by electromagnetic forces, and may be dealt with by the methods already described. The shape of the curve of fig. 3 suggests that the initial stage of growth may be extremely rapid, but we must remember that the figure refers to a long cylinder rather than to a small nucleus of approximately spherical shape. The nucleus will suffer from demagnetizing effects which will tend to increase the field strength at the magnetic equator, and decrease it at the poles. Thus the tendency will be for the most rapid expansion to occur along the magnetic lines of force, so that a long superconducting spindle will be formed, which will then expand, if it lies at the centre of the cylinder, after the manner of fig. 3 (the direction of the time axis being of course reversed).

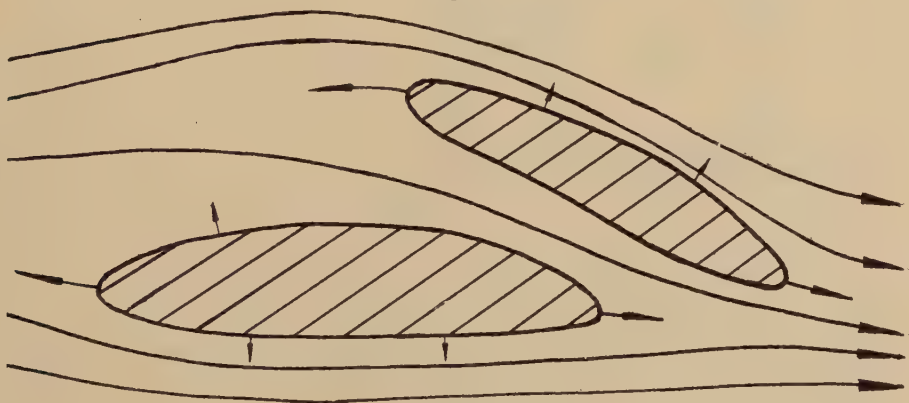
IV. THE ESTABLISHMENT OF THE INTERMEDIATE STATE.

So far we have considered only examples of the phase transition in a specimen of zero demagnetizing coefficient, for which ideally the complete transition should be possible at the field strength H_c . In bodies of more complicated shape, such as ellipsoids, the transition in an increasing field can occur reversibly only if it is spread out over a range of field strengths, and in this range, as has been shown directly by Meshkovsky and Shalnikov (1947), the material is split into alternate regions of superconducting and normal material, a condition known as the Intermediate State. A theory of the equilibrium configuration of the intermediate state has been given by Landau (1943), and in this section we shall attempt a qualitative explanation of the mechanism whereby it is established. Let us consider a plane parallel slab of normal material in a magnetic field whose direction is perpendicular to the surface of the slab. As the field strength is reduced below H_c there will be an initial stage during which nothing occurs, the material remaining in the supercooled normal state. Eventually, however, we may expect a nucleus of superconducting material to be formed, which will, in accordance with the ideas of the previous section, expand rapidly along the field lines, and not so rapidly in a radial direction, until at last a spindle will extend right across the specimen. This spindle should continue to expand radially until it takes up such a shape that over the whole interface between the normal and superconducting regions (though not at the boundary of the specimen) the field strength is just equal to H_c . This represents a stable configuration of the spindle. Now, during the whole of this process, the magnetic field has been expelled from the normal region, so that the field strength is never reduced below the value which it had before the nucleus was formed. There seems to be no reason therefore why any more spindles should be formed in the material until a chance fluctuation results in the creation of another nucleus. Nevertheless, it is observed in all studies of the intermediate state that once the supercooled phase has broken down there is a sudden transition into an intermediate

state filling the whole of the specimen. It appears as though the production of one nucleus leads to a chain of nucleation processes, enabling superconducting regions to be formed through the entire specimen.

It is possible that the explanation of this effect is to be found in the imperfections of the specimen, such as grain boundaries or local strains, which lead to an irregular jerky growth of a superconducting spindle. If as the spindle grows along the magnetic lines it meets an impediment, so that its progress is momentarily retarded, demagnetizing effects will lead to a decrease of the field strength at its forward end. Just in front of the impeded spindle there will then be a region which is particularly favourable for the production of a new nucleus. Once a new nucleus has formed, further growth should proceed roughly as indicated in fig. 4, in which the arrows attached to the spindles represent the directions favourable to further growth. If any magnetic lines are trapped between the spindles,

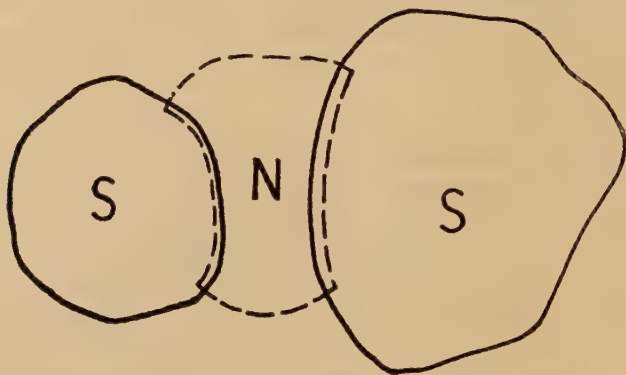
Fig. 4.



as will always happen unless the second nucleus is created exactly in front of the first spindle, there will arise electromagnetic forces which will effectively prevent coalescence of the two spindles. An adequate treatment of this problem, even by approximate methods, is difficult, but a qualitative argument should make the general principles clear. Consider the two spindles, S , shown in cross-section in fig. 5; for coalescence to occur it is necessary that the magnetic flux in the intervening region shall be ejected. Now the circuit shown dotted in the diagram may be regarded as an inductance, whose magnitude is determined by the area of the flux-containing region, N , of normal metal included in the circuit. As the spindles expand, L will vary approximately as the separation of the spindles. The resistance of the circuit will be determined by the length of the path in normal material between the spindles, and thus will also be roughly proportional to their separation. As the spindles expand, the time constant L/R for ejection of flux will therefore stay roughly constant, and the rate at which they approach one another will diminish continuously.

At the same time there is no such retardation of the growth in those directions in which there are no neighbouring spindles, so that coalescence is unlikely to occur before the spindles have expanded to such a size that the field strength in the normal environs has everywhere attained a value close to H_c , and further growth is impossible. We may therefore picture the establishment of the intermediate state as a spreading chain of nucleations, arising from the jerky growth of the initial and subsequent nuclei, until in the end the whole specimen is filled with superconducting spindles, and the intervening normal regions are permeated by a magnetic field of strength near H_c .

Fig. 5.



V. DESTRUCTION OF SUPERCONDUCTIVITY BY A CURRENT.

As a last problem in the kinetics of phase boundary movement we shall consider the destruction of superconductivity in a thin cylindrical wire carrying a current. According to Silsbee's (1918) hypothesis, when the current reaches the value $\frac{1}{2}aH_c$, a being the radius of the wire, the field at the surface attains the critical value, and at this point resistance appears. Silsbee's hypothesis has been found by Scott (1948) to give a precise description of the critical current for destruction of superconductivity, even in very thin wires where the effects of surface energy between the phases might be expected to play some part. The behaviour of wires carrying currents greater than critical has been studied theoretically by F. London (1937), who has shown that over a certain range of currents an intermediate state must be set up. This is observed in practice, the resistance being only partially restored at the critical current, and then rising slowly to the normal value with increase of the current. There is not complete agreement between the experimental behaviour and London's prediction, and this is attributed by Scott to the effects of surface energy. This aspect of the problem will not concern us here, since we shall consider only the kinetic problem of the transition from the superconducting to the intermediate state, without any discussion of the exact nature of the intermediate state itself.

The problem differs from that of the cylinder in a longitudinal field, discussed in Section III, in that here we have a longitudinal current and a circulating magnetic field. We shall see that this profoundly modifies the results of the calculation. Let us at first neglect the possibility that during the movement of the phase boundary there is a return to the superconducting state of those regions of normal material which find themselves in a field less than H_c ; that is, we shall suppose that at any instant during the transition the wire consists of an inner cylindrical core of superconducting material, of radius r_0 , surrounded by a sheath of normal material. If the current is such as to produce a field at the surface of the wire of strength $H_c(1+p)$, then the total current must be

$$I_{tot} = \frac{1}{2} a H_c (1+p),$$

of which the part carried by the superconducting core

$$I_s = \frac{1}{2} r_0 H_c,$$

in order to maintain a field strength H_c at the phase boundary. The remaining current will be carried in the normal metal,

$$I_n = \frac{1}{2} H_c \{a(1+p) - r_0\}. \quad (5)$$

We shall now use the approximate method of solution which assumes the field strength in the normal metal to have everywhere the value H_c . It immediately follows that the electric field induced by a shrinkage of the core may be written in the form $E = -H_c \frac{dr_0}{dt}$, at all points in the normal metal. There is thus a uniform current density in the normal metal, $J_n = -\sigma H_c \frac{dr_0}{dt}$, and the total current carried by the normal metal,

$$I_n = -\pi \sigma H_c (a^2 - r_0^2) \frac{dr_0}{dt}.$$

Comparing this result with equation (5) we have an expression for the rate of shrinkage,

$$-\frac{dr_0}{dt} = \frac{a(1+p) - r_0}{2\pi\sigma(a^2 - r_0^2)}.$$

Integrating this equation, we find an expression for the time taken for the superconducting core to shrink to a radius r_0 ,

$$t/2\pi\sigma a^2 = \frac{3}{2} + p - \rho(1+p + \frac{1}{2}\rho) + p(2+p) \log \frac{p}{1+p-\rho},$$

where

$$\rho = r_0/a. \quad (6)$$

In particular, the time taken for the superconducting core to disappear entirely is given by

$$t_0/2\pi\sigma a^2 = \frac{3}{2} + p + p(2+p) \log \frac{p}{1+p}.$$

The detailed form of this function is of little validity, as we shall see when we come to discuss the justification for the approximate method of solution, but its interest lies in the fact that if we put p equal to zero, we find a finite value, $3\pi\sigma a^2$, for t_0 . Thus the calculation predicts that, in contrast to the behaviour of a cylinder in a longitudinal field, as soon as the current reaches the critical value, the transition may be completed in a finite time. For wires of the size used in experiments on the destruction of superconductivity by a current this time is very short. If the radius of the wire is $\frac{1}{4}$ mm., and the material pure tin, the transition should be completed in 12 milliseconds.

So far it has been assumed that the transition consists solely of the shrinkage of the superconducting core until it finally disappears, but this cannot be the complete solution to the problem, since London has shown that stability is only achieved by the formation of an intermediate state. In order to understand how the intermediate state is set up, we must look into the magnetic field distribution in the normal metal. The field strength has been taken as H_c everywhere, but this assumption is not justified. In fact, if we now use our result that the current density is uniform in the normal metal to calculate the magnetic field, we find that when $p=0$ the field varies with radius according to the law

$$H(r) = H_c \frac{ar_0 + r^2}{r(a + r_0)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Thus H takes the value H_c when $r=r_0$ or a , in accordance with the boundary conditions, but is less than H_c at all intermediate radii. The minimum value of H occurs at a radius $\sqrt{ar_0}$, and at this point

$$H_{\min} = H_c \frac{2\sqrt{ar_0}}{a + r_0}.$$

As the core contracts, then, there are left behind regions of normal metal where the field strength is less than H_c , and when $r_0 \ll a$, there are regions near the centre of the wire where the field is very low. Presumably it is here that nucleation may occur for the establishment of the intermediate state, but no attempt has been made to carry the argument further, as it is difficult to conjecture the configuration of normal and superconducting regions which is stable in a wire carrying a current.

The manner in which the magnetic field is decreased below H_c in the normal metal suggests that the currents induced by the shrinkage of the core may have been overestimated, and hence the rate of shrinkage underestimated. A second approximation may be calculated by starting from the assumption that the field in the normal metal is given by equation (7), and it is found that in the final stage of collapse of the core the rate is indeed greater than was found in the first approximation. But the calculation is laborious, and the final result is complicated and mathematically not very well-behaved, so that it seems hardly worth while attempting any further stages of approximation.

The physical reason for the difference in behaviour of a cylinder when carrying a current and when subjected to a longitudinal field may be seen by considering the stability of the system under critical conditions. In the latter case, when the field strength is H_c , any diameter of superconducting material is stable, and the system may be said to be in neutral equilibrium. On the other hand, if we pass a critical current through the cylinder, and then suppose a normal sheath to be formed, eventually all the current will redistribute itself on the superconducting core of smaller diameter, producing there a field greater than H_c . A wire carrying the critical current is thus in unstable equilibrium, and as soon as any portion of the surface becomes normal there is a tendency for the surface current density in that region to increase and precipitate the transition into the normal state. This is probably the reason why Silsbee's hypothesis applies very well even in extremely thin wires, of the order of diameter of those which show effects of surface energy in transverse magnetic fields. The surface energy is unlikely to be great enough to counteract the instability of a wire carrying the critical current, though, of course, it will play a part in determining the configuration of normal and superconducting regions in the intermediate state, and hence the way in which the resistance approaches the normal value once the critical current has been exceeded. Such an hypothesis would be consistent with the observations of Scott.

I should like to thank Mr. T. E. Faber and Dr. B. Serin for interesting discussions on some of the points raised in this paper.

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XXII. *Programming a Computer for Playing Chess* *.

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1. INTRODUCTION.

THIS paper is concerned with the problem of constructing a computing routine or "program" for a modern general purpose computer which will enable it to play chess. Although perhaps of no practical importance, the question is of theoretical interest, and it is hoped that a satisfactory solution of this problem will act as a wedge in attacking other problems of a similar nature and of greater significance. Some possibilities in this direction are:—

- (1) Machines for designing filters, equalizers, etc.
- (2) Machines for designing relay and switching circuits.
- (3) Machines which will handle routing of telephone calls based on the individual circumstances rather than by fixed patterns.
- (4) Machines for performing symbolic (non-numerical) mathematical operations.
- (5) Machines capable of translating from one language to another.
- (6) Machines for making strategic decisions in simplified military operations.
- (7) Machines capable of orchestrating a melody.
- (8) Machines capable of logical deduction.

It is believed that all of these and many other devices of a similar nature are possible developments in the immediate future. The techniques developed for modern electronic and relay type computers make them not only theoretical possibilities, but in several cases worthy of serious consideration from the economic point of view.

Machines of this general type are an extension over the ordinary use of numerical computers in several ways. First, the entities dealt with are not primarily numbers, but rather chess positions, circuits, mathematical expressions, words, etc. Second, the proper procedure involves general principles, something of the nature of judgment, and considerable trial and error, rather than a strict, unalterable computing process. Finally, the solutions of these problems are not merely right or wrong but have a continuous range of "quality" from the best down to the worst. We might be satisfied with a machine that designed good filters even though they were not always the best possible.

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† Communicated by the Author.

The chess machine is an ideal one to start with, since : (1) the problem is sharply defined both in allowed operations (the moves) and in the ultimate goal (checkmate); (2) it is neither so simple as to be trivial nor too difficult for satisfactory solution; (3) chess is generally considered to require "thinking" for skilful play; a solution of this problem will force us either to admit the possibility of mechanized thinking or to further restrict our concept of "thinking"; (4) the discrete structure of chess fits well into the digital nature of modern computers.

There is already a considerable literature on the subject of chess-playing machines. During the late 18th and early 19th centuries, the Maelzel Chess Automaton, a device invented by von Kempelen, was exhibited widely as a chess-playing machine. A number of papers appeared at the time, including an analytical essay by Edgar Allan Poe (entitled Maelzel's Chess Player) purporting to explain its operation. Most of these writers concluded, quite correctly, that the Automaton was operated by a concealed human chess-master; the arguments leading to this conclusion, however, were frequently fallacious. Poe assumes, for example, that it is as easy to design a machine which will invariably win as one which wins occasionally, and argues that since the Automaton was not invincible it was therefore operated by a human, a clear *non séquitur*. For a complete account of the history and method of operation of the Automaton, the reader is referred to a series of articles by Harkness and Battell in *Chess Review*, 1947.

A more honest attempt to design a chess-playing machine was made in 1914 by Torrès y Quévedo, who constructed a device which played an end game of king and rook against king (Vigneron, 1914). The machine played the side with king and rook and would force checkmate in a few moves however its human opponent played. Since an explicit set of rules can be given for making satisfactory moves in such an end game, the problem is relatively simple, but the idea was quite advanced for that period.

The thesis we will develop is that modern general purpose computers can be used to play a tolerably good game of chess by the use of a suitable computing routine or "program". While the approach given here is believed fundamentally sound, it will be evident that much further experimental and theoretical work remains to be done.

2. GENERAL CONSIDERATIONS.

A chess "position" may be defined to include the following data :—

- (1) A statement of the positions of all pieces on the board.
- (2) A statement of which side, White or Black, has the move.
- (3) A statement as to whether the kings and rooks have moved. This is important since by moving a rook, for example, the right to castle on that side is forfeited.

(4) A statement of, say, the last move. This will determine whether a possible *en passant* capture is legal, since this privilege is forfeited after one move.

(5) A statement of the number of moves made since the last pawn move or capture. This is important because of the 50 move drawing rule. For simplicity, we will ignore the rule of draw after three repetitions of a position.

In chess there is no chance element apart from the original choice of which player has the first move. This is in contrast with card games, backgammon, etc. Furthermore, in chess each of the two opponents has "perfect information" at each move as to all previous moves (in contrast with Kriegspiel, for example). These two facts imply (von Neumann and Morgenstern, 1944) that any given position of the chess pieces must be either:—

(1) A won position for White. That is, White can force a win, however Black defends.

(2) A draw position. White can force at least a draw, however Black plays, and likewise Black can force at least a draw, however White plays. If both sides play correctly the game will end in a draw.

(3) A won position for Black. Black can force a win, however White plays.

This is, for practical purposes, of the nature of an existence theorem. No practical method is known for determining to which of the three categories a general position belongs. If there were chess would lose most of its interest as a game. One could determine whether the initial position is won, drawn, or lost for White and the outcome of a game between opponents knowing the method would be fully determined at the choice of the first move. Supposing the initial position a draw (as suggested by empirical evidence from master games*) every game would end in a draw.

It is interesting that a slight change in the rules of chess gives a game for which it is provable that White has at least a draw in the initial position. Suppose the rules the same as those of chess except that a player is not forced to move a piece at his turn to play, but may, if he chooses, "pass". Then we can prove as a theorem that White can at least draw by proper play. For in the initial position either he has a winning move or not. If so, let him make this move. If not, let him pass. Black is now faced with essentially the same position that White had before, because of the mirror symmetry of the initial position †. Since White had no winning move before, Black has none now. Hence, Black at best can draw. Therefore, in either case White can at least draw.

* The world championship match between Capablanca and Alekhine ended with the score Alekhine 6, Capablanca 3, drawn 25.

† The fact that the number of moves remaining before a draw is called by the 50-move rule has decreased does not affect this argument.

In some games there is a simple *evaluation function* $f(P)$ which can be applied to a position P and whose value determines to which category (won, lost, etc.) the position P belongs. In the game of Nim (Hardy and Wright, 1938), for example, this can be determined by writing the number of matches in each pile in binary notation. These numbers are arranged in a column (as though to add them). If the number of ones in each column is even, the position is lost for the player about to move, otherwise won.

If such an evaluation function $f(P)$ can be found for a game it is easy to design a machine capable of perfect play. It would never lose or draw a won position and never lose a drawn position and if the opponent ever made a mistake the machine would capitalize on it. This could be done as follows: Suppose

$$\begin{aligned} f(P) &= 1 \text{ for a won position,} \\ f(P) &= 0 \text{ for a drawn position,} \\ f(P) &= -1 \text{ for a lost position.} \end{aligned}$$

At the machine's turn to move it calculates $f(P)$ for the various positions obtained from the present position by each possible move that can be made. It chooses that move (or one of the set) giving the maximum value to f . In the case of Nim where such a function $f(P)$ is known, a machine has actually been constructed which plays a perfect game*.

With chess it is possible, *in principle*, to play a perfect game or construct a machine to do so as follows: One considers in a given position all possible moves, then all moves for the opponent, etc., to the end of the game (in each variation). The end must occur, by the rules of the game, after a finite number of moves† (remembering the 50 move drawing rule). Each of these variations ends in win, loss or draw. By working backward from the end one can determine whether there is a forced win, the position is a draw or is lost. It is easy to show, however, that even with the high computing speeds available in electronic calculators this computation is impractical. In typical chess positions there will be of the order of 30 legal moves. The number holds fairly constant until the game is nearly finished as shown in Fig. 1. This graph was constructed from data given by De Groot, who averaged the number of legal moves in a large number of master games (De Groot, 1946, *a*). Thus a move for White and then one for Black gives about 10^3 possibilities. A typical game lasts about 40 moves to resignation of one party. This is conservative for our calculation since the machine should calculate out to checkmate, not resignation. However, even at this figure there will be

* Condon, Tawney and Derr, U.S. Patent 2,215,544. The "Nimotron" based on this patent was built and exhibited by Westinghouse at the 1938 New York World's Fair.

† The longest possible chess game is 6350 moves, allowing 50 moves between each pawn move or capture. The longest tournament game on record between masters lasted 168 moves, and the shortest four moves. (Chernev, *Curious Chess Facts*, The Black Knight Press, 1937.)

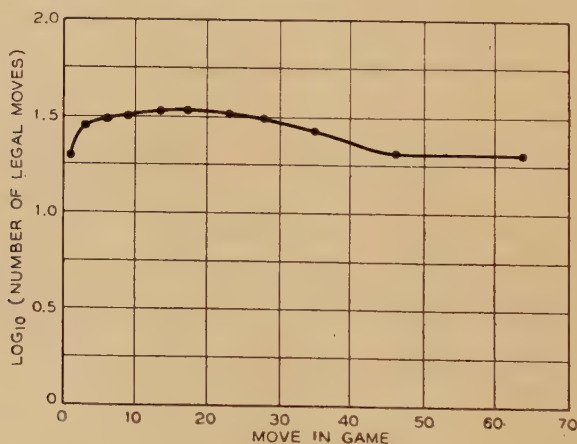
10^{120} variations to be calculated from the initial position. A machine operating at the rate of one variation per micro-microsecond would require over 10^{90} years to calculate its first move!

Another (equally impractical) method is to have a "dictionary" of all possible positions of the chess pieces. For each possible position there is an entry giving the correct move (either calculated by the above process or supplied by a chess master). At the machine's turn to move it merely looks up the position and makes the indicated move. The number of possible positions, of the general order of $64! / 32! 8! 2! 6!$, or roughly 10^{43} , naturally makes such a design unfeasible.

It is clear then that the problem is not that of designing a machine to play perfect chess (which is quite impractical) nor one which merely plays legal chess (which is trivial). We would like it to play a skilful game, perhaps comparable to that of a good human player.

A *strategy* for chess may be described as a process for choosing a move in any given position. If the process always chooses the same move in

Fig. 1.



the same position the strategy is known in the theory of games as a "pure" strategy. If the process involves statistical elements and does not always result in the same choice it is a "mixed" strategy. The following are simple examples of strategies:—

(1) Number the possible legal moves in the position P, according to some standard procedure. Choose the first on the list. This is a pure strategy.

(2) Number the legal moves and choose one at random from the list. This is a mixed strategy.

Both of these, of course, are extremely poor strategies, making no attempt to select good moves. Our problem is to develop a tolerably good strategy for selecting the move to be made.

3. APPROXIMATE EVALUATING FUNCTIONS.

Although in chess there is no known simple and exact evaluating function $f(P)$, and probably never will be because of the arbitrary and complicated nature of the rules of the game, it is still possible to perform an approximate evaluation of a position. Any good chess player must, in fact, be able to perform such a position evaluation. Evaluations are based on the general structure of the position, the number and kind of Black and White pieces, pawn formation, mobility, etc. These evaluations are not perfect, but the stronger the player the better his evaluations. Most of the maxims and principles of correct play are really assertions about evaluating positions, for example :—

(1) The relative values of queen, rook, bishop, knight and pawn are about 9, 5, 3, 3, 1, respectively. Thus other things being equal (!) if we add the numbers of pieces for the two sides with these coefficients, the side with the largest total has the better position.

(2) Rooks should be placed on open files. This is part of a more general principle that the side with the greater mobility, other things equal, has the better game.

(3) Backward, isolated and doubled pawns are weak.

(4) An exposed king is a weakness (until the end game).

These and similar principles are only generalizations from empirical evidence of numerous games, and only have a kind of statistical validity. Probably any chess principle can be contradicted by particular counter examples. However, from these principles one can construct a crude evaluation function. The following is an example :—

$$f(P) = 200(K - K') + 9(Q - Q') + 5(R - R') + 3(B - B' + N - N') + (P - P') \\ - \cdot 5(D - D' + S - S' + I - I') + \cdot 1(M - M') + \dots$$

in which :—

K, Q, R, B, N, P are the number of White kings, queens, rooks, bishops, knights and pawns on the board.

D, S, I are doubled, backward and isolated White pawns.

M = White mobility (measured, say, as the number of legal moves available to White).

Primed letters are the similar quantities for Black.

The coefficients $\cdot 5$ and $\cdot 1$ are merely the writer's rough estimate. Furthermore, there are many other terms that should be included*. The formula is given only for illustrative purposes. Checkmate has been artificially included here by giving the king the large value 200 (anything greater than the maximum of all other terms would do).

It may be noted that this approximate evaluation $f(P)$ has a more or less continuous range of possible values, while with an exact evaluation there are only three possible values. This is as it should be. In practical

* See Appendix I.

play a position may be an "easy win" if a player is, for example, a queen ahead, or a very difficult win with only a pawn advantage. In the former case there are many ways to win while in the latter exact play is required, and a single mistake often destroys the advantage. The unlimited intellects assumed in the theory of games, on the other hand, never make a mistake and the smallest winning advantage is as good as mate in one. A game between two such mental giants, Mr. A and Mr. B, would proceed as follows. They sit down at the chessboard, draw for colours, and then survey the pieces for a moment. Then either

- (1) Mr. A says, "I resign" or
- (2) Mr. B says, "I resign" or
- (3) Mr. A says, "I offer a draw," and Mr. B replies, "I accept."

4. STRATEGY BASED ON AN EVALUATION FUNCTION.

A very important point about the simple type of evaluation function given above (and general principles of chess) is that they can only be applied in relatively quiescent positions. For example, in an exchange of queens White plays, say, $Q \times Q$ ($\times$ = captures) and Black will reply $P \times Q$. It would be absurd to calculate the function $f(P)$ after $Q \times Q$ while White is, for a moment, a queen ahead, since Black will immediately recover it. More generally it is meaningless to calculate an evaluation of the general type given above during the course of a combination or a series of exchanges.

More terms could be added to $f(P)$ to account for exchanges in progress, but it appears that combinations, and forced variations in general, are better accounted for by examination of specific variations. This is, in fact, the way chess players calculate. A certain number of variations are investigated move by move until a more or less quiescent position is reached and at this point something of the nature of an evaluation is applied to the resulting position. The player chooses the variation leading to the highest evaluation for him when the opponent is assumed to be playing to reduce this evaluation.

The process can be described mathematically. We omit at first the fact that $f(P)$ should only be applied in quiescent positions. A strategy of play based on $f(P)$ and operating one move deep is the following. Let $M_1, M_2, M_3, \dots, M_s$ be the moves that can be made in position P and let M_1P, M_2P , etc. denote symbolically the resulting positions when M_1, M_2 , etc. are applied to P . Then one chooses the M_m which maximizes $f(M_mP)$.

A deeper strategy would consider the opponent's replies. Let $M_{i1}, M_{i2}, \dots, M_{is}$ be the possible answers by Black, if White chooses move M_i . Black should play to *minimize* $f(P)$. Furthermore, his choice occurs *after* White's move. Thus, if White plays M_i Black may be assumed to play the M_{ij} such that

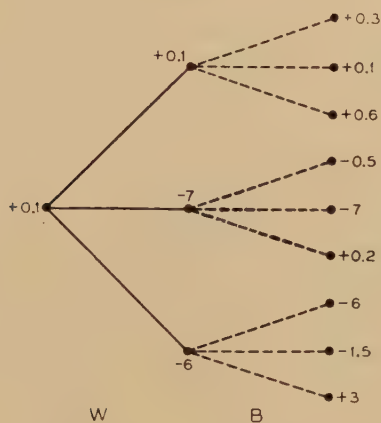
$$f(M_{ij}M_iP)$$

is a *minimum*. White should play his first move such that f is a maximum after Black chooses his best reply. Therefore, White should play to maximize on M_i the quantity

$$\min_{M_{ij}} f(M_{ij} M_i P).$$

The mathematical process involved is shown for a simple case in Fig. 2. The point at the left represents the position being considered. It is assumed that there are three possible moves for White, indicated by the three solid lines, and if any of these is made there are three possible moves for Black, indicated by the dashed lines. The possible positions after a White and Black move are then the nine points on the right, and the numbers are the evaluations for these positions. Minimizing on the upper three gives $+1$ which is the resulting value if White chooses the

Fig. 2.



upper variation and Black replies with his best move. Similarly, the second and third moves lead to values of -7 and -6 . Maximizing on White's move, we obtain $+1$ with the upper move as White's best choice.

In a similar way a two-move strategy (based on considering all variations out to 2 moves) is given by

$$\begin{array}{ccccccc} \text{Max} & \text{Min} & \text{Max} & \text{Min} & f(M_{ijkl} & M_{ijk} & M_{ij} & M_i & P) \\ M_i & M_{ij} & M_{ijk} & M_{ijkl} & & & & & \end{array} \quad (1)$$

The order of maximizing and minimizing this function is important. It derives from the fact that the choices of moves occur in a definite order.

A machine operating on this strategy at the two-move level would first calculate all variations out to two moves (for each side) and the resulting positions. The evaluations $f(P)$ are calculated for each of these positions. Fixing all but the last Black move, this last is varied and the move chosen which minimizes f . This is Black's assumed last move in

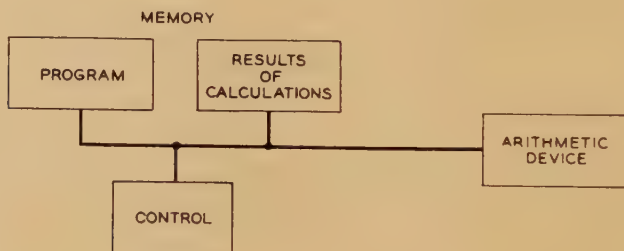
the variation in question. Another move for White's second move is chosen and the process repeated for Black's second move. This is done for each second White move and the one chosen giving the largest final f (after Black's best assumed reply in each case). In this way White's second move in each variation is determined. Continuing in this way the machine works back to the present position and the best first White move. This move is then played. This process generalizes in the obvious way for any number of moves.

A strategy of this sort, in which all variations are considered out to a definite number of moves and the move then determined from a formula such as (1) will be called a type A strategy. The type A strategy has certain basic weaknesses, which we will discuss later, but is conceptually simple, and we will first show how a computer can be programmed for such a strategy.

5. PROGRAMMING A GENERAL PURPOSE COMPUTER FOR A TYPE A STRATEGY.

We assume a large-scale digital computer, indicated schematically in Fig. 3, with the following properties:—

Fig. 3.



(1) There is a large internal memory for storing numbers. The memory is divided into a number of boxes each capable of holding, say, a ten-digit number. Each box is assigned a "box number".

(2) There is an arithmetic organ which can perform the elementary operations of addition, multiplication, etc.

(3) The computer operates under the control of a "program". The program consists of a sequence of elementary "orders". A typical order is A 372, 451, 133. This means, extract the contents of box 372 and of box 451, add these numbers, and put the sum in box 133. Another type of order involves a decision, for example, C 291, 118, 345. This tells the machine to compare the contents of box 291 and 118. If the first is larger the machine goes on to the next order in the program. If not, it takes its next order from box 345. This type of order enables the machine to choose from alternative procedures, depending on the results of previous calculations. It is assumed that orders are available for transferring numbers, the arithmetic operations, and decisions.

Our problem is to represent chess as numbers and operations on numbers, and to reduce the strategy decided upon to a sequence of computer orders. We will not carry this out in detail but only outline the programs. As a colleague puts it, the final program for a computer must be written in words of one microsyllable.

The rather Procrustean tactics of forcing chess into an arithmetic computer are dictated by economic considerations. Ideally, we would like to design a special computer for chess containing, in place of the arithmetic organ, a "chess organ" specifically designed to perform the simple chess calculations. Although a large improvement in speed of operation would undoubtedly result, the initial cost of computers seems to prohibit such a possibility. It is planned, however, to experiment with a simple strategy on one of the numerical computers now being constructed.

Fig. 4.

BLACK							
70	71	72	73	74	75	76	77
60	61	62	63	64	65	66	67
50	51	52	53	54	55	56	57
40	41	42	43	44	45	46	47
30	31	32	33	34	35	36	37
20	21	22	23	24	25	26	27
10	11	12	13	14	15	16	17
00	01	02	03	04	05	06	07
WHITE							

CODE FOR PIECES						
P	N	B	R	Q	K	
WHITE	1	2	3	4	5	6
BLACK	-1	-2	-3	-4	-5	-6
0 = EMPTY SQUARE						

CODE FOR MOVE	
(OLD SQUARE, NEW SQUARE, NEW PIECE (IF PROMOTION))	
P - K4 → (14, 34, -)	
P - K8(Q) → (64, 74, 5)	

A game of chess can be divided into three phases, the opening, the middle game, and the end game. Different principles of play apply in the different phases. In the opening, which generally lasts for about ten moves, development of the pieces to good positions is the main objective. During the middle game tactics and combinations are predominant. This phase lasts until most of the pieces are exchanged, leaving only kings, pawns and perhaps one or two pieces on each side. The end game is mainly concerned with pawn promotion. Exact timing and such possibilities as "Zugzwang", stalemate, etc. become important.

Due to the difference in strategic aims, different programs should be used for the different phases of a game. We will be chiefly concerned with the middle game and will not consider the end game at all. There seems no reason, however, why an end game strategy cannot be designed and programmed equally well.

A square on a chessboard can be occupied in 13 different ways: either it is empty (0) or occupied by one of the six possible kinds of White pieces ($P=1$, $N=2$, $B=3$, $R=4$, $Q=5$, $K=6$) or one of the six possible Black pieces ($P=-1$, $N=-2$, $\dots$, $K=-6$). Thus, the state of a square is specified by giving an integer from -6 to $+6$. The 64 squares can be numbered according to a co-ordinate system as shown in Fig. 4. The position of all pieces is then given by a sequence of 64 numbers each lying between -6 and $+6$. A total of 256 bits (binary digits) is sufficient memory in this representation. Although not the most efficient encoding, it is a convenient one for calculation. One further number λ will be $+1$ or -1 according as it is White's or Black's move. A few more should be added for data relating to castling privileges (whether the White or Black kings and rooks have moved), and *en passant* captures (*e.g.*, a statement of the last move). We will neglect these, however. In this notation the starting chess position is given by:—

4, 2, 3, 5, 6, 3, 2, 4; 1, 1, 1, 1, 1, 1, 1, 1;
 0, 0, 0, 0, 0, 0, 0, 0; 0, 0, 0, 0, 0, 0, 0, 0;
 0, 0, 0, 0, 0, 0, 0, 0; 0, 0, 0, 0, 0, 0, 0, 0;
 $-1, -1, -1, -1, -1, -1, -1, -1$; $-4, -2, -3, -5, -6, -3, -2, -4$;
 $+1 (= \lambda)$.

A move (apart from castling and pawn promotion) can be specified by giving the original and final squares occupied by the moved piece. Each of these squares is a choice from 64, thus 6 binary digits each is sufficient, a total of 12 for the move. Thus the initial move $P-K4$ would be represented by 1, 4; 3, 4. To represent pawn promotion a set of three binary digits can be added specifying the piece that the pawn becomes. Castling is described by giving the king move (this being the only way the king can move two squares). Thus, a move is represented by (a, b, c) where a and b are squares and c specifies a piece in case of promotion.

The complete program for a type A strategy consists of nine sub-programs which we designate $T_0, T_1, \dots, T_8$ and a master program T_9 . The basic functions of these programs are as follows:—

T_0 —Makes move (a, b, c) in position P to obtain the resulting position.

T_1 —Makes a list of the possible moves of a pawn at square (x, y) in position P .

$T_2, \dots, T_6$ —Similarly for other types of pieces: knight, bishop, rook, queen and king.

T_7 —Makes list of all possible moves in a given position.

T_8 —Calculates the evaluating function $f(P)$ for a given position P .

T_9 —Master program; performs maximizing and minimizing calculation to determine proper move.

With a given position P and a move (a, b, c) in the internal memory of the machine it can make the move and obtain the resulting position by the following program T_0 .

(1) The square corresponding to number a in the position is located in the position memory.

(2) The number in this square x is extracted and replaced by 0 (empty).

(3) (a) If $x=1$, and the first co-ordinate of a is 6 (White pawn being promoted) or if $x=-1$, and the first co-ordinate of a is 1 (Black pawn being promoted), the number c is placed in square b (replacing whatever was there).

(b) If $x=6$ and $a-b=2$ (White castles, king side) 0 is placed in squares 04 and 07 and 6 and 4 in squares 06 and 05, respectively. Similarly for the cases $x=6$, $b-a=2$ (White castles, queen side) and $x=-6$, $a-b=\pm 2$ (Black castles, king or queen side).

(c) In all other cases, x is placed in square b .

(4) The sign of λ is changed.

For each type of piece there is a program for determining its possible moves. As a typical example the bishop program, T_3 , is briefly as follows. Let (x, y) be the co-ordinates of the square occupied by the bishop.

(1) Construct $(x+1, y+1)$ and read the contents u of this square in the position P.

(2) If $u=0$ (empty) list the move (x, y) , $(x+1, y+1)$ and start over with $(x+2, y+2)$ instead of $(x+1, y+1)$.

If λu is positive (own piece in the square) continue to 3.

If λu is negative (opponent's piece in the square) list the move and continue to 3.

If the square does not exist continue to 3.

(3) Construct $(x+1, y-1)$ and perform similar calculation.

(4) Similarly with $(x-1, y+1)$.

(5) Similarly with $(x-1, y-1)$.

By this program a list is constructed of the possible moves of a bishop in a given position P. Similar programs would list the moves of any other piece. There is considerable scope for opportunism in simplifying these programs; *e.g.*, the queen program, T_5 , can be a combination of the bishop and rook programs, T_3 and T_4 .

Using the piece programs $T_1 \dots T_6$ and a controlling program T_7 the machine can construct a list of *all* possible moves in any given position P. The controlling program T_7 is briefly as follows (omitting details):—

(1) Start at square 1,1 and extract contents x .

(2) If λx is positive start corresponding piece program T_x and when complete return to (1) adding 1 to square number. If λx is zero or negative, return to 1 adding 1 to square number.

(3) Test each of the listed moves for legality and discard those which are illegal. This is done by making each of the moves in the position P (by program T_0) and examining whether it leaves the king in check.

With the programs $T_0 \dots T_7$ it is possible for the machine to play legal chess, merely making a randomly chosen legal move at each turn to move. The level of play with such a strategy is unbelievably bad.* The writer played a few games against this random strategy and was able to checkmate generally in four or five moves (by fool's mate, etc.). The following game will illustrate the utter purposelessness of random play :—

	White (Random)	Black
(1)	P-KN3	P-K4
(2)	P-Q3	B-B4
(3)	B-Q2	Q-B3
(4)	N-QB3	Q $\times$ P mate

We now return to the strategy based on an evaluation $f(P)$. The program T_8 performs the function of evaluating a position according to the agreed-upon $f(P)$. This can be done by the obvious means of scanning the squares and adding the terms involved. It is not difficult to include terms such as doubled pawns, etc.

The final master program T_9 is needed to select the move according to the maximizing and minimizing process indicated above. On the basis of one move (for each side) T_9 works as follows :—

- (1) List the legal moves (by T_7) possible in the present position.
- (2) Take the first in the list and make this move by T_0 , giving position M_1P .
- (3) List the Black moves in M_1P .
- (4) Apply the first one giving $M_{11}M_1P$, and evaluate by T_8 .
- (5) Apply the second Black move M_{12} and evaluate.
- (6) Compare, and reject the move with the smaller evaluation.
- (7) Continue with the third Black move and compare with the retained value, etc.
- (8) When the Black moves are exhausted, one will be retained together with its evaluation. The process is now repeated with the second White move.
- (9) The final evaluations from these two computations are compared and the maximum retained.
- (10) This is continued with all White moves until the best is selected (*i. e.* the one remaining after all are tried). This is the move to be made.

* Although there is a finite probability, of the order of 10^{-75} , that random play would win a game from Botvinnik. Bad as random play is, there are even worse strategies which choose moves which actually *aid* the opponent. For example, White's strategy in the following game: 1. P-KB3, P-K4. 2. P-KN4, Q-R5 mate.

These programs are, of course, highly iterative. For that reason they should not require a great deal of program memory if efficiently worked out.

The internal memory for positions and temporary results of calculations when playing three moves deep can be estimated. Three positions should probably be remembered: the initial position, the next to the last, and the last position (now being evaluated). This requires some 800 bits. Furthermore, there are five lists of moves each requiring about $30 \times 12 = 360$ bits, a total of 1800. Finally, about 200 bits would cover the selections and evaluations up to the present calculation. Thus, some 3000 bits should suffice.

6. IMPROVEMENTS IN THE STRATEGY.

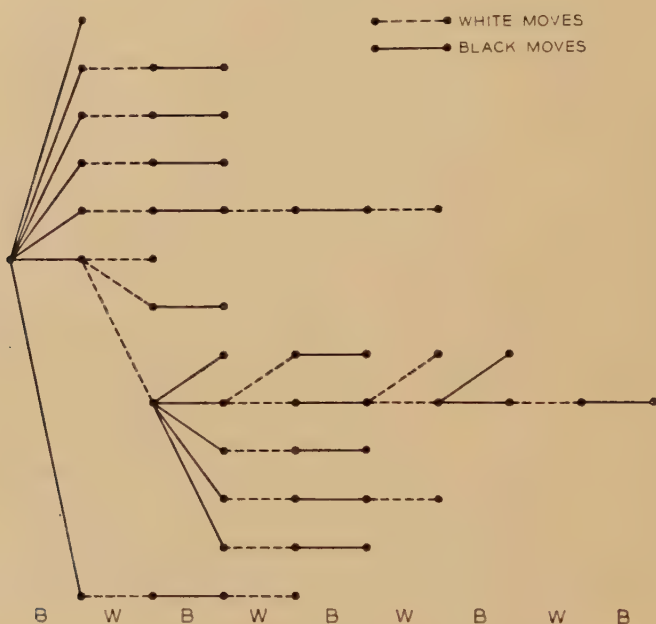
Unfortunately a machine operating according to this type A strategy would be both slow and a weak player. It would be slow since even if each position were evaluated in one microsecond (very optimistic) there are about 10^9 evaluations to be made after three moves (for each side). Thus, more than 16 minutes would be required for a move, or 10 hours for its half of a 40-move game.

It would be weak in playing skill because it is only seeing three moves deep and because we have not included any conditions about quiescent positions for evaluation. The machine is operating in an extremely inefficient fashion—it computes *all* variations to *exactly* three moves and then stops (even though it or the opponent be in check). A good human player examines only a few selected variations and carries these out to a reasonable stopping-point. A world champion can construct (at best) combinations say, 15 or 20 moves deep. Some variations given by Alekhine (“My Best Games of Chess 1924–1937”) are of this length. Of course, only a few variations are explored to any such depth. In amateur play variations are seldom examined more deeply than six or eight moves, and this only when the moves are of a highly forcing nature (with very limited possible replies). More generally, when there are few threats and forceful moves, most calculations are not deeper than one or two moves, with perhaps half-a-dozen forcing variations explored to three, four or five moves.

On this point a quotation from Reuben Fine (Fine 1942), a leading American master, is interesting: “Very often people have the idea that masters foresee everything or nearly everything; that when they played P–R3 on the thirteenth move they foresaw that this would be needed to provide a loophole for the king after the complications twenty moves later, or even that when they play 1 P–K4 they do it with the idea of preventing Kt–Q4 on Black’s twelfth turn, or they feel that everything is mathematically calculated down to the smirk when the Queen’s Rook Pawn queens one move ahead of the opponent’s King’s Knight’s Pawn. All this is, of course, pure fantasy. The best course to follow is to note the major consequences for two moves, but try to work out forced variations as they go.”

The amount of selection exercised by chess masters in examining possible variations has been studied experimentally by De Groot (1946, *b*). He showed various typical positions to chess masters and asked them to decide on the best move, describing aloud their analyses of the positions as they thought them through. In this manner the number and depth of the variations examined could be determined. Fig. 5 shows the result of one such experiment. In this case the chess master examined sixteen variations, ranging in depth from $1/2$ (one Black move) to $4\frac{1}{2}$ (five Black and four White) moves. The total number of positions considered was 44.

Fig. 5.



From these remarks it appears that to improve the speed and strength of play the machine must :—

(1) Examine forceful variations out as far as possible and evaluate only at reasonable positions, where some quasi-stability has been established.

(2) Select the variations to be explored by some process so that the machine does not waste its time in totally pointless variations.

A strategy with these two improvements will be called a type B strategy. It is not difficult to construct programs incorporating these features. For the first we define a function $g(P)$ of a position which determines whether approximate stability exists (no pieces *en prise*, etc.). A crude definition might be :

$$g(P) = \begin{cases} 1 & \text{if any piece is attacked by a piece of lower value, or by more} \\ & \text{pieces than defences or if any check exists on a square} \\ & \text{controlled by opponent.} \\ 0 & \text{otherwise.} \end{cases}$$

Using this function, variations could be explored until $g(P)=0$, always, however, going at least two moves and never more than say, 10.

The second improvement would require a function $h(P, M)$ to decide whether a move M in position P is worth exploring. It is important that this preliminary screening should *not* eliminate moves which merely look bad at first sight, for example, a move which puts a piece *en prise*; frequently such moves are actually very strong since the piece cannot be safely taken.

"Always give check, it may be mate" is tongue-in-cheek advice given to beginners aimed at their predilection for useless checks. "Always investigate a check, it may lead to mate" is sound advice for any player. A check is the most forceful type of move. The opponent's replies are highly limited—he can never answer by counter attack, for example. This means that a variation starting with a check can be more readily calculated than any other. Similarly captures, attacks on major pieces, threats of mate, etc. limit the opponent's replies and should be calculated whether the move looks good at first sight or not. Hence, $h(P, M)$ should be given large values for all forceful moves (checks, captures and attacking moves), for developing moves, medium values for defensive moves, and low values for other moves. In exploring a variation $h(P, M)$ would be calculated as the machine computes and would be used to select the variations considered. As it gets further into the variation the requirements on h are set higher so that fewer and fewer subvariations are examined. Thus, it would start considering every first move for itself, only the more forceful replies, etc. By this process its computing efficiency would be greatly improved.

It is believed that an electronic computer incorporating these two improvements in the program would play a fairly strong game, at speeds comparable to human speeds. It may be noted that a machine has several advantages over humans:—

- (1) High-speed operation in individual calculations.
- (2) Freedom from errors. The only errors will be due to deficiencies of the program while human players are continually guilty of very simple and obvious blunders.
- (3) Freedom from laziness. It is all too easy for a human player to make instinctive moves without proper analysis of the position.
- (4) Freedom from "nerves". Human players are prone to blunder due to over-confidence in "won" positions or defeatism and self-recrimination in "lost" positions.

These must be balanced against the flexibility, imagination and inductive and learning capacities of the human mind.

Incidentally, the person who designs the program can calculate the move that the machine will choose in any position, and thus in a sense can play an equally good game. In actual fact, however, the calculation would be impractical because of the time required. On a fair basis of comparison, giving the machine and the designer equal time to decide on a move, the machine might well play a stronger game.

7. VARIATIONS IN PLAY AND IN STYLE.

As described so far the machine once designed would always make the same move in the same position. If the opponent made the same moves this would always lead to the same game. It is desirable to avoid this, since if the opponent wins one game he could play the same variation and win continuously, due perhaps to some particular position arising in the variation where the machine chooses a very weak move.

One way to prevent this is to have a statistical element in the machine. Whenever there are two or more moves which are of nearly equal value according to the machine's calculations it chooses from them at random. In the same position a second time it may then choose another in the set.

The opening is another place where statistical variation can be introduced. It would seem desirable to have a number of the standard openings stored in a slow-speed memory in the machine. Perhaps a few hundred would be satisfactory. For the first few moves (until either the opponent deviates from the "book" or the end of the stored variation is reached) the machine plays by memory. This is hardly "cheating" since that is the way chess masters play the opening.

It is interesting that the "style" of play of the machine can be changed very easily by altering some of the coefficients and numerical factors involved in the evaluation function and the other programs. By placing high values on positional weaknesses, etc. a positional-type player results. By more intensive examination of forced variations it becomes a combination player. Furthermore, the strength of the play can be easily adjusted by changing the depth of calculation and by omitting or adding terms to the evaluation function.

Finally we may note that a machine of this type will play "brilliantly" up to its limits. It will readily sacrifice a queen or other piece in order to gain more material later or to give checkmate provided the completion of the combination occurs within its computing limits.

The chief weakness is that the machine will not learn by its mistakes. The only way to improve its play is by improving the program. Some thought has been given to designing a program which is self-improving but, although it appears to be possible, the methods thought of so far do not seem to be very practical. One possibility is to have a higher level program which changes the terms and coefficients involved in the evaluation function depending on the results of games the machine has played. Small variations might be introduced in these terms and the values selected to give the greatest percentage of "wins".

8. ANOTHER TYPE OF STRATEGY.

The strategies described above do not, of course, exhaust the possibilities. In fact, there are undoubtedly others which are far more efficient in the use of available computing time on the machine. Even with the improvements we have discussed the above strategy gives an impression of relying too much on "brute force" calculations rather than on logical analysis of a position. It plays something like a beginner at chess who has been told some of the principles and is possessed of tremendous energy and accuracy for calculation but has no experience with the game. A chess master, on the other hand, has available knowledge of hundreds or perhaps thousands of standard situations, stock combinations, and common manœuvres which occur time and again in the game. There are, for example, the typical sacrifices of a knight at B7 or a bishop at R7, the standard mates such as the "Philidor Legacy", manœuvres based on pins, forks, discoveries, promotion, etc. In a given position he recognizes some similarity to a familiar situation and this directs his mental calculations along lines with greater probability of success.

There is no reason why a program based on such "type positions" could not be constructed. This would require, however, a rather formidable analysis of the game. Although there are various books analysing combination play and the middle game, they are written for human consumption, not for computing machines. It is possible to give a person one or two specific examples of a general situation and have him understand and apply the general principle involved. With a computer an exact and completely explicit characterization of the situation must be given with all limitations, special cases, etc. taken into account. We are inclined to believe, however, that if this were done a much more efficient program would result.

To program such a strategy we might suppose that any position in the machine is accompanied by a rather elaborate analysis of the tactical structure of the position suitably encoded. This analytical data will state that, for example, the Black knight at B3 is pinned by a bishop, that the White rook at K1 cannot leave the back rank because of a threatened mate on B8, that a White knight at R4 has no move, etc.; in short, all the facts to which a chess player would ascribe importance in analysing tactical possibilities. These data would be supplied by a program and would be continually changed and kept up-to-date as the game progressed. The analytical data would be used to trigger various other programs depending on the particular nature of the position. A pinned piece should be attacked. If a rook must guard the back rank it cannot guard the pawn in front of it, etc. The machine obtains in this manner suggestions of plausible moves to investigate.

It is not being suggested that we should design the strategy in our own image. Rather it should be matched to the capacities and weaknesses of the computer. The computer is strong in speed and accuracy and weak in analytical ability and recognition. Hence, it should make more

use of brutal calculations than humans, but with possible variations increasing by a factor of 10^3 every move, a little selection goes a long way toward improving blind trial and error.

ACKNOWLEDGMENT.

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APPENDIX.

THE EVALUATION FUNCTION FOR CHESS.

The evaluation function $f(P)$ should take into account the "long term" advantages and disadvantages of a position, *i. e.* effects which may be expected to persist over a number of moves longer than individual variations are calculated. Thus the evaluation is mainly concerned with positional or strategic considerations rather than combinatorial or tactical ones. Of course there is no sharp line of division; many features of a position are on the borderline. It appears, however, that the following might properly be included in $f(P)$:—

- (1) Material advantage (difference in total material).
- (2) Pawn formation :
 - (a) Backward, isolated and doubled pawns.
 - (b) Relative control of centre (pawns at K4, Q4, B4).
 - (c) Weakness of pawns near king (*e. g.* advanced KNP).
 - (d) Pawns on opposite colour squares from bishop.
 - (e) Passed pawns.
- (3) Positions of pieces :
 - (a) Advanced knight (at K5, Q5, B5, K6, Q6, B6), especially if protected by pawn and free from pawn attack.
 - (b) Rook on open file, or semi-open file.
 - (c) Rook on seventh rank.
 - (d) Doubled rooks.
- (4) Commitments, attacks and options :
 - (a) Pieces which are required for guarding functions and, therefore, committed and with limited mobility.
 - (b) Attacks on pieces which give one player an option of exchanging.
 - (c) Attacks on squares adjacent to king.
 - (d) Pins. We mean here immobilizing pins where the pinned piece is of value not greater than the pinning piece: for example, a knight pinned by a bishop.
- (5) Mobility.

These factors will apply in the middle game : during the opening and end game different principles must be used. The relative values to be given each of the above quantities is open to considerable debate, and should be determined by some experimental procedure. There are also numerous other factors which may well be worth inclusion. The more violent tactical weapons, such as discovered checks, forks and pins by a piece of lower value are omitted since they are best accounted for by the examination of specific variations.

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XXIII. *Propagation of Electromagnetic Disturbances along a Thin Wire in a Horizontally Stratified Medium.*

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SUMMARY.

It is shown that for a thin wire buried at finite depth in a semi-infinite homogeneous medium and subject to a disturbance at a given frequency, there exists an exponential attenuation of the current in the wire with a propagation constant equal to that of the medium. This result is extended to a medium with a number of layers, and to a wire lying in an interface.

LIST OF PRINCIPAL SYMBOLS.

All quantities are in M.K.S. rationalized units (Stratton 1941). Time-wise variation is as $e^{j\omega t}$, this factor being omitted for brevity.

x, y, z	} = Corresponding cartesian and cylindrical polar coordinate systems.
r, θ, z	
σ	= Conductivity of medium.
μ	= Permeability of medium.
ϵ	= Permittivity of medium.
ω	= $2\pi \times$ frequency.
γ	= Propagation constant of medium. $[j\omega\mu(\sigma + j\omega\epsilon)]^{\frac{1}{2}}$.
$\Gamma(\omega)$	= Propagation constant of disturbance along the wire, at angular pulsation ω .
a	= Radius of wire.
h	= Height of wire above interface.
$V(x)$	= Potential at x with respect to that at a point at infinity.
$E(x)$	= Potential gradient at x .
$\vec{E}$	= Electric intensity vector; components E_x, E_y, E_z .
$\vec{H}$	= Magnetic intensity vector; components H_x, H_y, H_z .
$\vec{\Pi}$	= Hertzian vector potential; components Π_x, Π_y, Π_z .
$I(x)$	= Current in wire at x .
$\bar{g}$	= Complex leakage conductance per unit length.
$\bar{Z}$	= Complex internal impedance per unit length.
$J_n(z)$	= Bessel function of first kind of order n (McLachlan 1941).
$K_n(z)$	= Modified Bessel function of second kind of order n^2 .
u, f, λ	= Arbitrary variables.
α	= $(u^2 + \gamma^2)^{\frac{1}{2}}$.
p	= $2\pi jf$.

The convention is used throughout that the root sign, as in $(u^2 + \gamma^2)^{\frac{1}{2}}$, or in $(\lambda^2)^{\frac{1}{2}}$, shall mean that root whose real part is non-negative. Subscripts 0 and 1 refer to the upper and lower media respectively.

* Communicated by Dr. S. Whitehead.

§ 1. INTRODUCTION.

THE problem of propagation along a buried wire has received attention in recent years in connection with the lightning protection of overhead transmission lines by buried wires (counterpoise) (Sunde 1940), and in the design of protective methods for buried power and telephone cables (Sunde 1945).

The propagation of surges can be calculated from steady state propagation by the use of operational methods such as the Fourier Integral (Campbell and Foster), so that it is necessary to solve only the latter case. Assume a current at a given frequency to be injected into the wire at a given point ; as it proceeds along the wire it leaks to earth, and an associated system of travelling electromagnetic waves is set up in accordance with Maxwellian theory. If the wire, which is assumed infinite in both directions, is buried at infinite depth in an homogeneous medium, the attenuation of the current follows the law :

$$I(x)=\pm\frac{1}{2}I(0)\exp\{-\gamma|x|\}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where γ is the propagation constant of the medium. The problem is relatively simple compared with that discussed here, because of the inherent symmetry of the system. The simpler result has been extended to the case of a wire lying in the interface between two media of different characteristics, assuming the conductivity and permittivity of the upper medium (air) are negligible, and the result is

$$I(x)=\pm\frac{1}{2}I(0)\exp\{-\gamma|x|/\sqrt{2}\}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where γ is the propagation constant of the lower medium (earth). It has been further suggested (Sunde 1945) that in this case, for moderate depths of burial of the wire, propagation is given approximately by equation (2). However, the method of proof of these results is open to criticism on mathematical grounds.

This paper deals with the general question of propagation along a wire buried at a finite depth in an earth consisting of a number of horizontal stratifications or layers, of which the air can be considered as one, of individual characteristics σ, μ, ϵ , each or all of which may differ between the various layers. It is shown that for thin wires there exists an exponential mode of propagation of the form $\exp\{-\Gamma|x|\}$, where the propagation constant Γ is determined as follows :

(1) If the wire is totally immersed in a layer of constant γ , that is, if it is more than a few diameters away from the nearest interface, then $\Gamma=\gamma$.

(2) If the wire lies at the interface between layers m and $m+1$

$$\Gamma=\left(\frac{\mu_{m+1}\gamma_m^2+\mu_m\gamma_{m+1}^2}{\mu_m+\mu_{m+1}}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

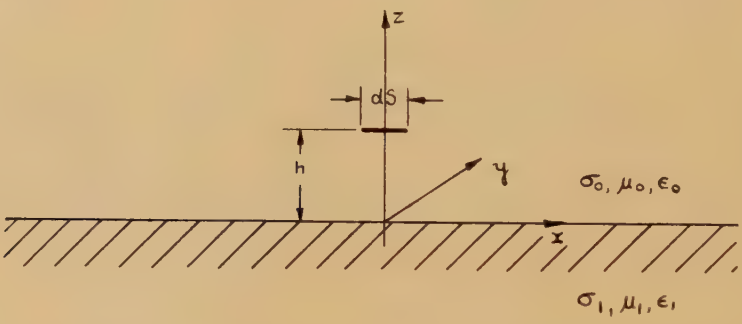
Writing $\gamma_m=0, \mu_m=\mu_{m+1}$, equation (2) follows.

The proof as given below is for the case of a two-layer medium, but from its methods, and the nature of the result, the above generalization is immediate (§ 5). It follows the methods used by Sunde (1936, 1940), extending his results and putting them on a rigorous mathematical foundation.

§ 2. THE FIELD OF A DOUBLET.

The doublet is supposed at height h above the interface between two semi-infinite homogeneous media specified by $\sigma_0, \mu_0, \epsilon_0$ and $\sigma_1, \mu_1, \epsilon_1$, as shown in fig. 1. It is of length ds , in the direction of the x axis with its mid-point on the z axis at $(0, 0, h)$, and carrying unit current. If the

Fig. 1.



two media were identical its field would be completely specified by a Hertzian vector potential everywhere in the x direction of magnitude

$$\pi'_{0x} = A \int_0^\infty \frac{\exp\{-\alpha_0|z-h|\}}{\alpha_0} u J_0(ru) du, \quad (4)$$

where

$$A = \frac{j\omega\mu_0 ds}{4\pi\gamma_0^2}$$

and (r, θ, z) is the associated cylindrical polar coordinate system. When the media are as shown (4) gives the primary field in the upper region, and the complete field is obtained by superimposing suitable secondary or reflection fields in the upper and lower media. These reflection fields are similarly obtainable from Hertzian vector potentials, and symmetry with respect to the x - z plane restricts such vectors to the x and z directions, so that the complete field can be specified by

$$\pi_{0x} = \pi_{0x}' + \pi_{0x}'', \quad \pi_{0z}$$
$$\pi_{1x} \quad , \quad \pi_{1z}$$

in the two media. The secondary components are everywhere finite and continuous, since the only permissible singularity is the doublet allowed for in π'_{0x} . The field intensities are given by

$$\left. \begin{aligned} H_x &= \frac{\gamma^2}{j\omega\mu} \frac{\partial \pi_z}{\partial y}, \\ H_y &= \frac{\gamma^2}{j\omega\mu} \left(\frac{\partial \pi_x}{\partial z} - \frac{\partial \pi_z}{\partial x} \right), \\ H_z &= \frac{\gamma^2}{j\omega\mu} \frac{\partial \pi_x}{\partial y}, \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\left. \begin{aligned} E_x &= -\gamma^2 \pi_x + \frac{\partial}{\partial x} \left(\frac{\partial \pi_x}{\partial x} + \frac{\partial \pi_z}{\partial z} \right), \\ E_y &= \frac{\partial}{\partial y} \left(\frac{\partial \pi_x}{\partial x} + \frac{\partial \pi_z}{\partial z} \right), \\ E_z &= -\gamma^2 \pi_z + \frac{\partial}{\partial z} \left(\frac{\partial \pi_x}{\partial x} + \frac{\partial \pi_z}{\partial z} \right), \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The secondary vector components satisfy Poisson's equation

$$\frac{\partial^2 \pi}{\partial x^2} + \frac{\partial^2 \pi}{\partial y^2} + \frac{\partial^2 \pi}{\partial z^2} = \gamma^2 \pi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and the appropriate general solution is (Stratton 1941, § 6.9; eq. (55))

$$\pi = \sum_{n=-\infty}^{n=+\infty} \exp \{nj\theta\} \int_0^\infty [g_n(u) \exp \{-\alpha z\} + f_n(u) \exp \{\alpha z\}] u J_n(ru) du. \quad . \quad (8)$$

The boundary conditions at $z=0$ consist of the continuity of the tangential (x, y) components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$. From the nature of the expansions (4) and (8), these conditions may be differentiated or integrated with respect to x and y in the plane $z=0$, and since the potentials must vanish at infinity in this plane the "functional constants" of integration must vanish. The conditions may therefore be taken as

$$z=0$$

$$\frac{\gamma_0^2}{j\omega\mu_0} \pi_{0z} = \frac{\gamma_1^2}{j\omega\mu_1} \pi_{1z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\frac{\gamma_0^2}{j\omega\mu_0} \frac{\partial \pi_{0x}}{\partial z} = \frac{\gamma_1^2}{j\omega\mu_1} \frac{\partial \pi_{1x}}{\partial z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\frac{\partial \pi_{0x}}{\partial x} + \frac{\partial \pi_{0z}}{\partial z} = \frac{\partial \pi_{1x}}{\partial x} + \frac{\partial \pi_{1z}}{\partial z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\gamma_0^2 \pi_{0x} = \gamma_1^2 \pi_{1x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Taking the appropriate forms of (5), namely

$$\pi''_{0x} = \sum_{n=-\infty}^{n=+\infty} \exp \{nj\theta\} \int_0^\infty f_{0n}(u) \exp \{-\alpha_0 z\} u J_n(ru) du, \quad . \quad (13)$$

etc., together with (4), equations (10) and (12) give

$$\pi''_{0x} = \int_0^\infty \frac{A \exp\{-\alpha_0 h\}}{\alpha_0} \cdot \frac{\mu_1 \alpha_0 - \mu_0 \alpha_1}{\mu_1 \alpha_0 + \mu_0 \alpha_1} \exp\{-\alpha_0 z\} u J_0(ru) du, \quad (14)$$

$$\pi_{1x} = \int_0^\infty \frac{2\gamma_0^2}{\gamma_1^2} A \exp\{-\alpha_0 h\} \cdot \frac{\mu_1}{\mu_1 \alpha_0 + \mu_0 \alpha_1} \exp\{\alpha_1 z\} u J_0(ru) du, \quad (15)$$

Noting that

$$\frac{\partial}{\partial x} J_0(ru) = -\cos \theta u J_1(ru) = -\frac{1}{2}(\exp\{j\theta\} + \exp\{-j\theta\})u J_1(ru), \quad (16)$$

equations (9) and (11) yield

$$\pi_{0z} = -\frac{\partial}{\partial x} \int_0^\infty \frac{2(\gamma_0^2 - \gamma_1^2)\mu_1 \mu_0 A \exp\{-\alpha_0 h\}}{(\mu_1 \alpha_0 + \mu_0 \alpha_1)(\gamma_0^2 \mu_1 \alpha_1 + \gamma_1^2 \mu_0 \alpha_0)} \exp\{-\alpha_0 z\} u J_0(ru) du, \quad (17)$$

$$\pi_{1z} = -\frac{\partial}{\partial x} \int_0^\infty \frac{2\gamma_0^2(\gamma_0^2 - \gamma_1^2)\mu_1^2 A \exp\{-\alpha_0 h\}}{\gamma_1^2(\mu_1 \alpha_0 + \mu_0 \alpha_1)(\gamma_0^2 \mu_1 \alpha_1 + \gamma_1^2 \mu_0 \alpha_0)} \exp\{\alpha_0 z\} u J_0(ru) du. \quad (18)$$

The electric intensity in the x direction in the plane $z=h$ is given by

$$E_x = ds[P(r) + \frac{\partial^2}{\partial x^2} Q(r)], \quad (19)$$

where

$$P(r) = \frac{-j\omega\mu_0}{4\pi} \int_0^\infty \frac{1}{\alpha_0} \left[1 + \frac{\mu_1 \alpha_0 - \mu_0 \alpha_1}{\mu_1 \alpha_0 + \mu_0 \alpha_1} \exp\{-2\alpha_0 h\} \right] u J_0(ru) du, \quad (20)$$

$$Q(r) = -\frac{1}{\gamma_0^2} P(r) + \frac{(\gamma_0^2 - \gamma_1^2)j\omega\mu_1 \mu_0^2}{2\pi\gamma_0^2} \int_0^\infty \frac{\alpha_0 \exp\{-2\alpha_0 h\} u J_0(ru) du}{(\mu_1 \alpha_0 + \mu_0 \alpha_1)(\gamma_0^2 \mu_1 \alpha_1 + \gamma_1^2 \mu_0 \alpha_0)}. \quad (21)$$

In the general case of a stratified earth with a number of layers, additional terms must be added to the components of the vector potential to represent higher order reflection fields, and such terms must again satisfy Poisson's equation and be free from singularities in the layer or layers containing the dipole. The effect of such terms on the solution is discussed in §5.

§3. PROPAGATION ALONG AN INFINITE WIRE (Sunde 1936).

The wire is supposed embedded in the upper medium at height h from the interface with its axis along the x axis, and the origin of coordinates is taken at the point on it opposite the centre of the applied disturbance (*e. g.* if current is fed directly into the wire the origin is the point of feed). The wire is of radius a and internal impedance $\bar{Z}$ (complex) per unit length which is supposed constant along its length for a given frequency. To account for imperfect contact it is supposed surrounded by a shell of negligible thickness and of complex leakage conductance $\bar{g}$ per unit length, again supposed constant along its length for a given frequency.

Suppose the applied disturbance gives rise to an electric intensity $E^0(x)$ along the surface of the wire.

Let $V_w(x)$ =potential of wire at (x, a, h)

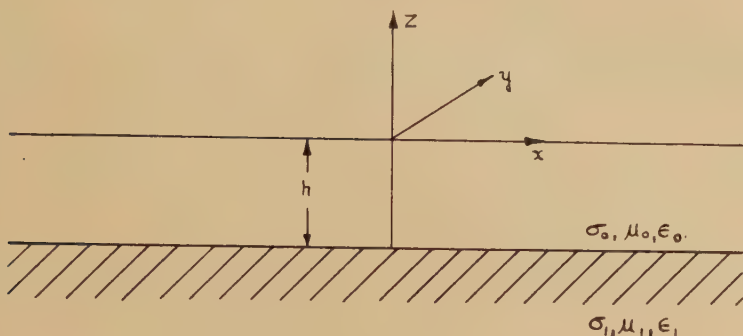
$V_m(x)$ =potential of the medium at an adjacent point.

$I(x)$ =total current in the wire at x .

Then

$$V_m(x) - V_w(x) = \bar{g}^{-1} \frac{dI(x)}{dx}. \quad (22)$$

Fig. 2.



Let $E_w(x)$ =potential gradient along the wire at (x, a, h)

$E_m(x)$ =potential gradient in the medium at an adjacent point in the x direction.

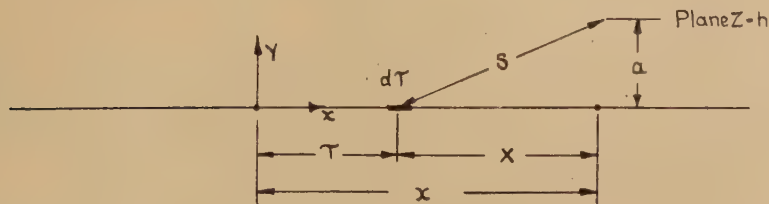
Then

$$E_m(x) - E_w(x) = -\bar{g}^{-1} \frac{d^2I(x)}{dx^2}. \quad (23)$$

Now by definition of $\bar{Z}$ if the wire is sufficiently thin for $E_w(x)$ to be constant over its surface, $E_w(x) = \bar{Z}I(x)$. Also $E_m(x)$ can be split into two parts, $E^0(x)$ the applied gradient and $E_c(x)$ the gradient due to currents in the surrounding media. Hence

$$\bar{g}^{-1} \frac{d^2I(x)}{dx^2} - \bar{Z}I(x) + E_c(x) = -E^0(x). \quad (24)$$

Fig. 3.



It remains to express $E_c(x)$ in terms of $I(x)$. With the aid of the results obtained in §2 $E_c(x)$ can be calculated in the plane $z=h$ as follows: referring to fig. 3 consider the effect of a current element at point $(\tau, 0, h)$,

on the electric field at (x, a, h) , and let $X=x-\tau$, $s=(X^2+a^2)^{\frac{1}{2}}$. Then

$$\delta E_c(x)=I(\tau) d\tau \left[P(s) + \frac{\partial^2}{\partial X^2} Q(s) \right], \quad . \quad . \quad . \quad . \quad (25)$$

where the P and Q functions are given by (20) and (21).

$$\begin{aligned} \therefore E_c(x) &= \int_{-\infty}^{+\infty} P(s) I(\tau) d\tau + \int_{-\infty}^{+\infty} I(\tau) \frac{\partial^2}{\partial X^2} Q(s) d\tau \\ &= T_1 + T_2 \text{ (say).} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26) \end{aligned}$$

The current can be expressed by a Fourier Integral as

$$I(x) = \int_{-\infty}^{+\infty} F(\lambda) \exp\{j\lambda x\} d\lambda, \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where λ is real on the path of integration. Then

$$\begin{aligned} * T_1 &= \int_{-\infty}^{+\infty} d\tau P(s) \int_{-\infty}^{+\infty} F(\lambda) \exp\{j\lambda\tau\} d\lambda \quad . \quad . \quad . \quad . \quad (28) \\ &= \int_{-\infty}^{+\infty} d\lambda F(\lambda) \int_{-\infty}^{+\infty} P(s) \exp\{j\lambda\tau\} d\tau \end{aligned}$$

writing

$$\begin{aligned} \tau &= x + v, \quad b = (v^2 + a^2)^{\frac{1}{2}}, \\ T_1 &= \int_{-\infty}^{+\infty} d\lambda F(\lambda) \exp\{j\lambda x\} \int_{-\infty}^{+\infty} P(b) \exp\{j\lambda v\} dv. \quad . \quad . \quad (29) \end{aligned}$$

In a similar manner, noting that $\frac{\partial}{\partial X} = \frac{\partial}{\partial x}$ it follows that

$$T_2 = - \int_{-\infty}^{+\infty} d\lambda F(\lambda) \exp\{j\lambda x\} \lambda^2 \int_{-\infty}^{+\infty} Q(b) \exp\{j\lambda v\} dv, \quad . \quad (30)$$

writing

$$p(\lambda, a) = - \int_{-\infty}^{+\infty} P(b) \exp\{j\lambda v\} dv, \quad . \quad . \quad . \quad . \quad . \quad (31)$$

$$q(\lambda, a) = \int_{-\infty}^{+\infty} Q(b) \exp\{j\lambda v\} dv, \quad . \quad . \quad . \quad . \quad . \quad (32)$$

and expressing $E^0(x)$ as a Fourier integral

$$E^0(x) = \int_{-\infty}^{+\infty} h(\lambda) \exp\{j\lambda x\} d\lambda, \quad . \quad . \quad . \quad . \quad . \quad (33)$$

equation (24) becomes

$$\begin{aligned} & \int_{-\infty}^{+\infty} F(\lambda) \exp\{j\lambda x\} \{ \lambda^2 [\bar{g}^{-1} + q(\lambda, a)] + [\bar{Z} + p(\lambda, a)] \} d\lambda \\ &= \int_{-\infty}^{+\infty} h(\lambda) \exp\{j\lambda x\} d\lambda. \quad . \quad . \quad . \quad . \quad . \quad (34) \end{aligned}$$

* For a discussion of the inversion of this and other integrals see Appendix.

Writing

$$\Delta(\lambda) = \lambda^2 [\bar{g}^{-1} + q(\lambda, a)] + [\bar{Z} + p(\lambda, a)], \quad (35)$$

then

$$F(\lambda) = \frac{h(\lambda)}{\Delta(\lambda)}$$

or

$$I(x) = \int_{-\infty}^{+\infty} \frac{h(\lambda)}{\Delta(\lambda)} \exp \{j\lambda x\} d\lambda (36)$$

From equations (20) and (21), $P(b)$ and $Q(b)$ are integrals of the form

$$N(b) = \int_0^\infty F(u^2) u J_0(bu) du (37)$$

and hence $p(\lambda, a)$, $q(\lambda, a)$ are integrals of the form

$$G = \int_{-\infty}^{+\infty} N[(v^2 + a^2)^{\frac{1}{2}}] \exp \{j\lambda v\} dv,$$

where as previously remarked, λ is a real variable. In accordance with the notes in the Appendix it follows that

$$\begin{aligned} G &= \int_0^\infty du F(u^2) u \int_{-\infty}^{+\infty} \exp \{j\lambda v\} J_0[u(a^2 + v^2)^{\frac{1}{2}}] dv \\ &= \int_{|\lambda|}^\infty F(u^2) u \cdot \frac{2 \cos a(u^2 - \lambda^2)^{\frac{1}{2}}}{(u^2 - \lambda^2)^{\frac{1}{2}}} du \\ &= 2 \int_0^\infty F(u^2 + \lambda^2) \cos au du \\ &= \int_{-\infty}^{+\infty} \exp \{jau\} F(u^2 + \lambda^2) du. \quad (38) \end{aligned}$$

(These transformations follow from Campbell and Foster, pair 915.4.)
Hence

$$\begin{aligned} p(\lambda, a) &= \frac{j\omega\mu_0}{4\pi} \int_{-\infty}^{+\infty} \exp \{jau\} (u^2 + \lambda^2 + \gamma_0^2)^{-\frac{1}{2}} \\ &\quad \times \left\{ 1 + \frac{\mu_1(u^2 + \lambda^2 + \gamma_0^2)^{\frac{1}{2}} - \mu_0(u^2 + \lambda^2 + \gamma_1^2)^{\frac{1}{2}}}{\mu_1(u^2 + \lambda^2 + \gamma_0^2)^{\frac{1}{2}} + \mu_0(u^2 + \lambda^2 + \gamma_1^2)^{\frac{1}{2}}} \exp \{-2h(u^2 + \lambda^2 + \gamma_0^2)^{\frac{1}{2}}\} \right\} du \\ &\quad (39) \end{aligned}$$

with a similar expression for $q(\lambda, a)$, these expressions holding for real values of λ . Consider them now as functions of a complex variable λ , then as $|\lambda| \rightarrow \infty$

$$\begin{aligned} p(\lambda, a) &\rightarrow \frac{j\omega\mu_0}{2\pi} \int_{-\infty}^{+\infty} \exp \{jau\} (u^2 + \lambda^2)^{\frac{1}{2}} \left[1 + \frac{\mu_1 - \mu_0}{\mu_1 + \mu_0} \exp \{-2h(u^2 + \lambda^2)^{\frac{1}{2}}\} \right] du \\ &= \frac{j\omega\mu_0}{2\pi} \left[K_0\{(\lambda^2)^{\frac{1}{2}} | a | \} + \frac{\mu_1 - \mu_0}{\mu_1 + \mu_0} K_0\{(\lambda^2)^{\frac{1}{2}} (a^2 + 4h^2)^{\frac{1}{2}}\} \right]. \quad (40) \end{aligned}$$

$$\rightarrow 0. \quad (41)$$

(See Campbell and Foster, pairs 868 and 558 and McLachlan, list of formulæ No. 111.)

Similarly $q(\lambda, a) \rightarrow 0$ as $\lambda \rightarrow 0$. If now $E^0(x)$ fulfils certain conditions (Whittaker and Watson 1940), equation (36) can be evaluated by the theory of residues. Suppose the poles of the numerator and zeros of the denominator are given by λ_i, λ_k ($i, k = 1, 2, 3 \dots$) where the integrand is now considered as a function of a complex variable λ , and i, k refer to the upper and lower halves of the λ plane respectively, it follows that

$$I(x) = 2\pi j \sum_i R_i \quad (x > 0) \quad . \quad . \quad . \quad . \quad . \quad (42)$$

$$= -2\pi j \sum_k R_k \quad (x < 0) \quad . \quad . \quad . \quad . \quad . \quad (43)$$

where R_i, R_k are the residues of the integrand at λ_i, λ_k respectively. It has been assumed that there are no real poles or roots, but the corresponding modifications are simple, and in any case it is shown below that it is not necessary in general to consider this possibility. (42) and (43) give the formal solutions of the problem and particular cases of interest are discussed in the next section.

§4. THE PRINCIPAL MODE OF PROPAGATION.

(4.1) Wire not in Interface.

In this case $h \neq 0$. Consider first the solutions corresponding to the zeros of the function $\Delta(\lambda)$, i. e. to the roots of

$$\lambda^2 = - \frac{\bar{Z} + p(\lambda, a)}{\bar{g}^{-1} + q(\lambda, a)} \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$p(\lambda, a)$ is given by (39), which can be written

$$p(\lambda, a) = \frac{j\omega\mu_0}{4\pi} \{ 2K_0[(\lambda^2 + \gamma_0^2)^{\frac{1}{2}} |a|] + T \}, \quad . \quad . \quad . \quad (45)$$

where

$$T = \int_{-\infty}^{+\infty} \frac{\mu_1(u^2 + \lambda^2 + \gamma_0^2)^{\frac{1}{2}} - \mu_0(u^2 + \lambda^2 + \gamma_1^2)^{\frac{1}{2}}}{\mu_1(u^2 + \lambda^2 + \gamma_0^2)^{\frac{1}{2}} + \mu_0(u^2 + \lambda^2 + \gamma_1^2)^{\frac{1}{2}}} \frac{\exp\{-2h(u^2 + \lambda^2 + \gamma_0^2)^{\frac{1}{2}}\}}{(u^2 + \lambda^2 + \gamma_0^2)^{\frac{1}{2}}} e^{jau} du.$$

Suppose a to decrease indefinitely; it is seen by the theory of residues that T remains finite, whilst $K_0[(\lambda^2 + \gamma_0^2)^{\frac{1}{2}} |a|] \rightarrow \infty$, and hence as $a \rightarrow 0, |p(\lambda, a)| \rightarrow \infty$. Returning to equations (20) and (21) it follows by an exactly similar argument that $q(\lambda, a) \rightarrow p(\lambda, a)/\gamma_0^2$, and supposing $\bar{Z}$ and $\bar{g}^{-1}$ finite it follows that the roots are given by

$$\lambda^2 \rightarrow -\gamma_0^2 \quad . \quad . \quad . \quad . \quad . \quad (46)$$

and the corresponding component of $I(x)$ is given by

$$I(x) \rightarrow I(0) \exp\{-\gamma_0 |x|\}, \quad . \quad . \quad . \quad . \quad . \quad (47)$$

where in this expression $I(0)$ as obtained by letting x tend to zero may be different according as x is +ve or -ve. For example, in the case of current fed into the wire at $x=0$, it is obvious by symmetry that $I(0+) = -I(0-)$. Furthermore, examination of the corresponding residues R_i and R_k shows that $h(\lambda)$ must have a singularity at $\lambda^2 = -\gamma_0^2$, and

this places an important restriction on the type of excitation which will cause a disturbance in the wire, though it is a restriction which is satisfied in many practical cases (Sunde 1936).

The remaining modes of propagation correspond to the poles of $h(\lambda)$ other than these, and it is seen that as $a \rightarrow 0$, $\lambda^2 \neq \gamma_0^2$, propagation is prevented by the infinity assumed by $\Delta(\lambda)$, so that for thin wires propagation will be exponential with a constant given by $\Gamma = \gamma_0$ (see § 1(1)).

(4.2) Wire in the Interface.

Similar arguments concerning the possible modes of propagation corresponding to the poles of $h(\lambda)$ at values of λ other than the roots of $\Delta(\lambda) = 0$ still apply, and it is only necessary to consider the roots of this equation, viz. (44). In this case $h = 0$ and (20) and (21) now give

$$P(b) = \frac{-j\omega\mu_0\mu_1}{2\pi} \int_0^\infty \frac{1}{\mu_0\alpha_1 + \mu_1\alpha_0} u J_0(bu) du \quad . \quad . \quad . \quad (48)$$

$$Q(b) = \frac{j\omega\mu_0\mu_1}{2\pi(\gamma_1\mu_0^2 - \gamma_0^2\mu_1^2)} \int_0^\infty \left[\frac{\mu_0^2 - \mu_1^2}{\mu_1\alpha_0 + \mu_0\alpha_1} - \frac{(\gamma_0^2 - \gamma_1^2)\mu_1\mu_0}{\gamma_0^2\mu_1\alpha_1 + \gamma_1^2\mu_0\alpha_0} \right] u J_0(bu) du. \quad (49)$$

Proceeding as above it is necessary to evaluate integrals of the form

$$G = \int_{-\infty}^{+\infty} \exp\{jau\} \frac{1}{\beta_0(u^2 + \lambda^2 + \gamma_0^2)^{\frac{1}{2}} + \beta_1(u^2 + \lambda^2 + \gamma_1^2)^{\frac{1}{2}}} du, \quad . \quad . \quad (50)$$

where β_0 and β_1 are complex constants. Now

$$G = \frac{1}{\beta_1^2 - \beta_0^2} \int_{-\infty}^{+\infty} \exp\{jau\} \left\{ \frac{\beta_1(u^2 + \lambda^2 + \gamma_1^2)^{\frac{1}{2}} - \beta_0(u^2 + \lambda^2 + \gamma_0^2)^{\frac{1}{2}}}{u^2 + \lambda^2 + \psi^2} \right\} du,$$

where

$$\psi^2 = \frac{\beta_1^2\gamma_1^2 - \beta_0^2\gamma_0^2}{\beta_1^2 - \beta_0^2}.$$

But

$$\int_{-\infty}^{+\infty} \exp\{jau\} \cdot \frac{(u^2 + \lambda^2 + \gamma^2)^{\frac{1}{2}}}{u^2 + \lambda^2 + \psi^2} du = 2\pi \int_{-\infty}^{+\infty} \exp\{2\pi jaf\} \frac{(\lambda^2 + \gamma^2 - p^2)^{\frac{1}{2}}}{\lambda^2 + \psi^2 - p^2},$$

where $p = 2\pi jf$, and it can be shown (Campbell and Foster, pairs 558, 438, 439, 202) that

$$\lim_{a \rightarrow 0} \int_{-\infty}^{+\infty} \exp\{2\pi jaf\} \frac{(\lambda^2 + \gamma^2 - p^2)^{\frac{1}{2}}}{\lambda^2 + \psi^2 - p^2} df = \lim_{a \rightarrow 0} \frac{1}{\pi} K_0[\lambda^2 + \gamma^2]^{\frac{1}{2}} |a| + \dots + \frac{\gamma^2 - \psi^2}{\pi(\lambda^2 + \psi^2)^{\frac{1}{2}}} \int_0^\infty \exp\{-x(\lambda^2 + \psi^2)^{\frac{1}{2}}\} K_0[(\lambda^2 + \gamma^2)^{\frac{1}{2}}x] dx, \quad . \quad . \quad (51)$$

and since the square root has a non-negative real part the second term is finite (Admiralty Computing Service). It follows that as $a \rightarrow 0$

$$p(\lambda, a) \rightarrow \frac{j\omega\mu_0\mu_1}{2\pi} \cdot \frac{2}{\mu_0^2 - \mu_1^2} \{\mu_0 K_0[a(\lambda^2 + \gamma_1^2)^{\frac{1}{2}}] - \mu_1 K_0[a(\lambda^2 + \gamma_0^2)^{\frac{1}{2}}]\} \quad . \quad (52)$$

$$q(\lambda, a) \rightarrow \frac{j\omega\mu_0\mu_1}{2\pi} \cdot \frac{2}{\gamma_1^2\mu_0^2 - \gamma_0^2\mu_1^2} \{[\mu_0 K_0\{a(\lambda^2 + \gamma_1^2)^{\frac{1}{2}}\} - \mu_1 K_0\{a(\lambda^2 + \gamma_0^2)^{\frac{1}{2}}\}] + \frac{(\gamma_0^2 - \gamma_1^2)\mu_1\mu_0}{\gamma_1^4\mu_0^2 - \gamma_0^4\mu_1^2} [\gamma_0^2\mu_1 K_0\{a(\lambda^2 + \gamma_1^2)^{\frac{1}{2}}\} - \gamma_1^2\mu_0 K_0\{a(\lambda^2 + \gamma_0^2)^{\frac{1}{2}}\}]\}. \quad . \quad . \quad (53)$$

APPENDIX.

ON THE INTERCHANGE IN THE ORDER OF INTEGRATION IN CERTAIN REPEATED INTEGRALS.

(a) In equations (28) and (38) repeated infinite integrals were transformed by interchanging the order of integration. This is always permissible if one pair of limits is finite, and the integrand is continuous in both variables within the limits of integration, but where both limits are infinite the argument breaks down, and it is only in very restricted circumstances that general theorems are available.

In the case of equation (38), where the integrand is a known function, a laborious proof of the legitimacy of the interchange can be constructed, and is outlined in (b) below. The justification of (28) must rest on less rigid arguments of a physical nature. If the integrands are functions which represent physically possible distributions of potential, and if the final answer must be that of a real physical problem, an answer obtained by interchanging the order of integration is likely to be correct if there is no reasonable alternative solution.

This presupposes some test of the validity of the solution adopted. An example is the equation

$$\begin{aligned} \int_0^\infty J_0(at) \cos(bt) dt &= (a^2 - b^2)^{-\frac{1}{2}} \text{ for } b < a \\ &= \infty \quad \text{,, } b = a \\ &= 0 \quad \text{,, } b > a \end{aligned}$$

an equation similar to those in the text. At $b=a$ two possibilities exist. If b approaches a from below, the limit value of infinity is obtained. If b approaches a from above the continuation value of the integral by a summation process is zero. The selection of the value at $b=a$ would depend on physical arguments, assuming that the final result must agree with other available evidence. Thus if the quantity could not be infinite then the representation is inexact and a special study of the physical conditions around $b=a$ would have to be made.

In the present instance the results obtained agree satisfactorily with the physical requirements of the problem and with the available experimental evidence, so that the procedure adopted is considered valid.

(b) In equation (38) the transformation

$$\begin{aligned} \int_0^\infty \cos \lambda v dv \int_0^\infty F(u^2) u J_0[u(v^2 + a^2)^{\frac{1}{2}}] du \\ = \int_0^\infty F(u^2) u du \int_0^\infty \cos \lambda v J_0[u(v^2 + a^2)^{\frac{1}{2}}] dv \quad \dots \quad (56) \end{aligned}$$

is used, in which $F(u^2)$ is any one of certain functions as described in the text. The inner integral on the L.H.S. is the electric intensity of a Hertzian dipole, so that from energy considerations it must be $L^2(0, \infty)$, and so its Fourier transform exists. The R.H.S. has been shown to

exist in the text. Writing unity for $F(u^2)u$ in (56) the truth of the modified equation is demonstrable by direct evaluation of the integrals concerned, so that is sufficient to prove (56) for $F(u^2)u - C$ where C is any constant. This must be done for the real and imaginary parts of $F(u^2)$, which can be any one of the following functions:

$$\begin{aligned} & (u^2 + \gamma_0^2)^{-\frac{1}{2}} \\ & \frac{\mu_1(u^2 + \gamma_0^2)^{\frac{1}{2}} - \mu_0(u^2 + \gamma_1^2)^{\frac{1}{2}}}{\mu_1(u^2 + \gamma_0^2)^{\frac{1}{2}} + \mu_0(u^2 + \gamma_1^2)^{\frac{1}{2}}} \cdot \frac{\exp \{-2h(u^2 + \gamma_0^2)^{\frac{1}{2}}\}}{(u^2 + \gamma_0^2)^{\frac{1}{2}}} \quad (h \geq 0) \\ & \frac{(u^2 + \gamma_0^2)^{\frac{1}{2}} \exp \{-2h(u^2 + \gamma_0^2)^{\frac{1}{2}}\}}{[\mu_1(u^2 + \gamma_0^2)^{\frac{1}{2}} + \mu_0(u^2 + \gamma_1^2)^{\frac{1}{2}}][\gamma_0^2 \mu_1(u^2 + \gamma_1^2)^{\frac{1}{2}} + \gamma_1^2 \mu_0(u^2 + \gamma_0^2)^{\frac{1}{2}}]} \quad (h > 0) \end{aligned}$$

Introducing the function

$$H(u) = \lim_{u \rightarrow \infty} \operatorname{Re}_{Im} \{u[F(u^2)u - C]\}$$

where in each case C is $\lim_{u \rightarrow \infty} F(u^2)u$, it can be shown that u_0 exists such that for $u > u_0$ $H(u)$ is a monotonic function increasing or decreasing to zero, and since the sign of $H(u)$ is immaterial the second case is assumed.

Now by using the asymptotic expansion of the Bessel function it can be shown that

$$\int_{x_1}^{x_2} du \int_{y_1}^{y_2} \frac{\cos \lambda v J_0[u(v^2 + a^2)^{\frac{1}{2}}]}{u} dv$$

is bounded for $x_1 > x_0$, $y_1 > y_0$ where $\infty \geq x_2 \geq x_1$, $\infty \geq y_2 \geq y_1$.

It follows by applying the second mean value theorem that

$$\left| \int_{x_1}^{x_2} \frac{H(u)}{u} du \int_{y_1}^{y_2} \cos \lambda v J_0[u(v^2 + a^2)^{\frac{1}{2}}] dv \right| < \epsilon$$

for all $x_2 \geq x_1 \geq x(\epsilon)$, $y_2 \geq y_1 \geq y(\epsilon)$ and the required result follows without further trouble.

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XXIV. *Quantal Aspects of Scientific Information.*

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SUMMARY.

This paper relates to the borderline linking experimental and theoretical physics with mathematical logic, and covers at several points ground which is common to the theory of communication. Its purpose is, therefore, to introduce in language as far as possible common to workers in all these fields, a quantitative approach to the analysis of scientific concepts and the design of experiments.

Many scientific concepts in different fields have a logically equivalent structure. One can abstract from them a logical form which is quite general, and takes on different particular meanings according to the context. Identification of this structure in precise terms leads to a clarification both of experimental principles and of fundamental relations in different fields of physics.

It is suggested that the most fundamental abstract scientific concept is quantal in its communicable aspects. It is defined as "Information-content" and has two features: the *a priori* or structural, and the *a posteriori* or quantitative. To each there corresponds a *quantum* of information, representing the minimal elementary proposition relating respectively to the structural and quantitative aspects of scientific statements.

Scientific statements are regarded as complexes of these elementary propositions. The information-content of a result can be completely represented by means of a vector in a multidimensional space, the dimensionality of the space and the square of the length of the vector indicating respectively the amounts of structural and quantitative information provided, while the orientation of the vector *specifies* the result. The information-content is shown to be fundamentally limited by the number of *conceptual units of space-time* devoted to the experiment, with obvious practical implications. Expressions are also derived measuring the amount of detail possible in a result under different conditions.

From the standpoint here adopted the various uncertainty-relations of physics appear basically as axioms expressing the quantal nature of communicable information, consequent on the use of logical forms; and the quantity *entropy plus information-content* appears as a fundamental

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invariant of a physical system. The present paper is, however, intended mainly to be explanatory and introductory, and consideration of more fundamental implications is postponed.

A glossary of less-familiar or new terms introduced is appended.

I. *The Formalism of Information Theory.*

§ 1. INTRODUCTION.

WHEN a physicist ventures into the domain of epistemology he is perhaps especially vulnerable to the charge of arbitrariness in his treatment and choice of definitions. This difficulty is particularly acute in connection with a theory of scientific information, for the brimstone flavour of metaphysics hangs over the subject, and adds zest to criticism. The present paper is an introductory account of an approach which seeks to reduce arbitrariness to a minimum by an appeal to logical first principles.

Experimentation abounds with indications that the everyday concepts of science are not the most fundamental. Each time that a compromise has to be struck, say, between the sensitivity and the response-time of a galvanometer, or the noise-level and band-width of an amplifier, or the resolving power and aperture of a microscope, one has an intuitive feeling that in each case some quantity is remaining constant behind all experimental manipulations—something more fundamental than either of the quantities in question. We say that “Nature cannot be cheated”; and examples of this principle recur throughout the realm of measurement, and not only in microphysics.

Is there not then a way of expressing scientific facts so that in any context a single universal principle can apply? Presumably in sufficiently fundamental terms such a principle should become obvious. This thought at least was the stimulus to the present investigation, which leads to the definition of an abstract concept common to all scientific statements, and responsible for their logical significance. In terms of this quantitative concept of *Information*, various semi-intuitive principles can be seen to have a precisely definable basis in a general axiom.

The establishment of a basic formalism for this purpose will occupy the first part of the paper. The second will deal in more detail with matters raised by the first, and will outline some of the practical implications of the theory. It is emphasized, however, that any examples given are intended mainly to help readers with different backgrounds to understand the context, and are not meant to anticipate the more thorough treatment of particular applications which it is hoped to give in later papers. In particular references to communication theory are designed mainly to establish the connection between the author's approach and that which has independently been so elegantly developed by various workers in that field (*e.g.* Gabor 1946, Shannon 1948, Wiener 1948).

§ 2. ACQUISITION OF SCIENTIFIC INFORMATION.

It is necessary, with some apology, to begin with a brief analysis of the nature of the scientific method and scientific propositions, in order to introduce our terms and arrive at the minimal form in which scientific facts can be expressed without loss of significance.

A scientific statement may be defined as a precise description of certain events pictured as populating a tract of a coordinate-space. Its essential purpose is the communication of information derived from an *experiment*—an activity in which events are classified. The acquisition of scientific *information* thus involves two distinct problems, each of which has given rise to a somewhat loose appropriation of the term in a technical sense.

Firstly, one must devise apparatus and/or prepare some system of classification, such that an adequate number of independent categories can be defined when describing the result. For example, if fluctuations varying in frequency between 1 and 100 per second are to be observed, the apparatus must be capable of responding in a time of the order of $1/100$ second, *i. e.* of giving 100 “independent” readings per second. Or again, if ten shades of colour are recognized as distinct modes of description (*i. e.* coordinate-values on a “colour-axis”) of the members of a population requiring to be classified, then ten columns must be provided in the observer’s notebook.

When a chain of apparatus is involved (including the observer) then the differentiating capacity of the least-discriminating link determines the number of independent categories in the result.

There is a sense in which this number, that is to say the number of independent dimensions or “degrees of freedom”, can be regarded as a measure of the information supplied by the experiment. (Its analogue in the case of communication-theory has been so defined by Gabor (1946)). It is, however, inadequate to represent completely the information derived from an experiment, as we shall see. Giving a more general connotation to a term coined by Gabor, we shall define it in due course as the *logon-content* of a result.

Secondly, the experiment must be performed, by using our apparatus (be it galvanometer, microscope, or eye-plus-notebook) to classify events in the chosen tract of coordinate-space. For instance we may record 100 independent values of a quantity as a function of time—the amount of that quantity which is associated with each of the 100 identifying-points provided by our method in the time-tract considered. We may divide a sample of a population into ten colour-groups, determining the fraction associated with each of the ten identifying points on the colour-axis.

Performance of the experiment thus results in the association of a number with each of the “labels”—categories or degrees of freedom—defined by the structure of the experimental method. The usual scientific statement, however, takes the form of an inference having a certain probability deduced from the experimental result. Thus we arrive

at a second common use of the term *information*, to signify the source of *confidence in a given number as representative* of the class identified by its label.

As we shall see, it is possible to give a precise numerical significance to this concept also. It is the essential *a posteriori* complement to the *a priori* logon content, for the complete representation of the information derived from an experiment.

So much for the context in which what follows is relevant. We must now begin at the other end, so to speak, by analysing the nature of the scientific statements made to convey the results of an experiment. In this analysis we shall find a basis, free of arbitrariness, for a quantal representation of scientific information.

§ 3. THE QUANTIZATION OF INFORMATION.

(a) *Elementary propositions.*

A scientific statement is a logical form based on *limited* data. As such it can be dissected ultimately into a pattern of elementary "atomic" propositions (Sheffer 1913, Wittgenstein 1922). Each atomic proposition states a fact so simple that it cannot be further decomposed, and hence has only the characteristic "true" or "false"; its existence is its only attribute. Each therefore owes its individuality only to its position in the logical pattern and may be regarded as a kind of link between two entities in the pattern. Ultimately the latter must make contact with the primitive sense-data in terms of which it acquires meaning, but this aspect of the subject does not concern us here. We shall treat as "elementary" the simplest propositions relating to the concepts of measurement and classification. Any debt so incurred to psychology is better settled out of court.

Ideally, a scientific statement is based entirely on *observable* evidence, and the ideal statement which would describe all the information supplied by a particular experiment is presumably reducible to a pattern of independent elementary propositions relating to observations. We should expect therefore to be able to define a measure of *information-content* corresponding to the number of such propositions substantiated by a given experiment. On this view we are led to define one kind of *unit* of information as *that which decides us to add one elementary proposition* to the "frame" (the pattern of propositions) which is logically sufficient to define the results observed.

(b) *Quantization of Information.*

The element of *decision* here appears to be fundamental, for it leads to a quite irreducible *quantization* of the action of scientific information. This is forced on us by our use of logical forms—by our apparent incapability of communicating information unambiguously in any other way. Our statements are rigorous; to speak of an "imperceptible change" in a statement would be meaningless; the minimum change

possible is the addition of one element to the logical pattern ; and a unit of information is simply that which causes us to make one such addition.

Assuming then that the formation of scientific statements is a quantal process, the next task is to identify our quanta in terms of the processes of experimentation studied in § 2. It is here that care is necessary to avoid confusing the two common uses of the term *information*. *Prior* information is presented by knowledge of the experimental procedure ; *posterior* information arises as the latter is carried out. The one defines the structure of the ultimate statement ; the other defines the amount of evidence which it subsumes. Each is quantal in its communicable aspects, though, of course, the respective " quanta " differ qualitatively. It will best serve our purpose to consider first *posterior* information.

§ 4. POSTERIOR OR METRICAL INFORMATION.

(a) *Scale-Units.*

The quantal character of posterior information arises from the way in which we describe a scientific measurement. A description of a result is basically a set of instructions enabling the reader to reproduce for himself a conceptual pattern representing the experience of the observer. The scientific discipline recognizes only one type of elementary experience as necessary and sufficient for this purpose ; the most elementary observational proposition asserts the existence of a *coincidence-relation* between two entities. On the other hand we define a *magnitude* by saying that it occupies a certain interval on a scale. Logically this *occupance-relation* between scale-interval and magnitude is a consequence of the existence of coincidence-relations between the ends of the " unknown " and two definable graduation-entities on the scale.

Now the fineness of graduation of a scale, whether physically marked or estimated, is always limited for reasons which will shortly be examined. The point at present is that for every observation there is a minimum separation between neighbouring graduation-entities, B_{n-1}, B_n, B_{n+1} , say, below which either we cannot define or cannot substantiate with probability greater than one-half, a proposition of the form : " A falls into $B_{n-1}-B_n$ and not into B_n-B_{n+1} ". This smallest interval we shall call the *scale-unit* appropriate to the observation.

(b) *Metron-content.*

With a scale marked in such " minimum meaningful intervals ", a magnitude can be specified by the number of intervals which it occupies. This number provides an index of information which we can describe as *metrical*, relating to *measurement*. The unit of metrical information, which we shall term a *metron*, is that which enables one elementary interval to be represented as occupied, in the logical pattern to be communicated. In other words each metron specifies one elementary occupance-relation.

Thus what we carry away from a measurement is basically an integer, the number of conceptually separate occupance-relations which have been specified. This integer we shall refer to as the *metron-content* of the result.

There are certain requirements which must be satisfied if the metron-content is to be equated to the number of atomic propositions justified by a given experimental sequence. It must be so defined that the metron-content of two similar but independent sequences is twice that of either alone. The metron-content of a result must be incapable of augmentation by purely logical manipulation, and all complete representations of a given result (*e.g.* in terms of a frequency-spectrum, or a time-function) should have the same metron-content.

If these conditions are satisfied, a result yielding a given metron-content provides a fixed number of logical elements out of which dependent or equivalent statements can be constructed. We can regard the making of these statements as a rearranging of the elements of the logical pattern. A suitable formalism to represent this process will be considered in § 6.

(c) *Proper scales.*

It is obvious that not all physical quantities can be represented on scales having a constant scale-unit. For example the square root of a measured quantity has presumably the same metron-content as the quantity itself, since the latter can be derived from it by a purely logical process; but the metron-content cannot be *proportional* to both quantities. Conceptual scales, which we may call *proper scales*, are not necessarily linear in terms of physical magnitude.

The proper scale of a quantity subject to random fluctuations is a good example of this fact. Fisher (1935) justifies his definition of statistical "information" (which is analogous to our metron-content) as a quantity proportional to the reciprocal of variance, by noting that variance depends inversely on the number of samples involved, and is a measure of the uncertainty with which a given sample can be regarded as representative. If the number of samples is multiplied by n , the information provided (assuming independence) should be n times as great. Since the same process reduces the variance by $1/n$, the information provided should therefore be proportional to its reciprocal.

The metron-content i of a single measurement of a randomly fluctuating quantity x having a probable error $\pm \frac{1}{2} \Delta x$ is therefore not $x/\Delta x$ but $i = x^2/(\Delta x)^2$, and the physical scale of x is divided into significant intervals which are non-uniform*. (The scale of x^2 , however, is uniformly divided.) This means that we can give a probability p of just $\frac{1}{2}$ to a statement in the form: x (or x^2) occupies i intervals. The figure of $\frac{1}{2}$ implies, not

* On the abstract conceptual scale of course only *order* and *number* are significant, so that all intervals are equivalent. It would be meaningless to say they were *equal*.

that we are ignorant of the magnitude of x , but that its terminal is as likely to lie within the i th interval as to fall outside it. It also implies that it would be unprofitable to employ a narrower interval in making our statement, for we should then be bound to make it in a form with $p < \frac{1}{2}$: more likely to be false than true.

(d) *Observation and measurement.*

In more complex cases there is a clearer distinction between *observation* and *measurement*. For instance we observe the position of a pointer, and hence measure a voltage. Now the precision of observation certainly sets an upper limit to that of measurement. Basically we observe the distance between the pointer and some reference mark (which must be a point at which another reading has been taken under standard conditions. It is possible to deduce the metron-content of the figure we obtain for this distance, and we could call it the metron-content of the *observation*, but it is not of fundamental significance. In principle it is the reproducibility of the pattern of events symbolized by the statement made which governs the metron-content of a measurement. The greater the metron-content the more detailed a statement can be made with a 50-50 chance of non-contradiction. Thus fluctuations in the readings may be the limiting factor. Indeed one may say that in a well-designed experiment this is always the case (*cf.* the operation of radio search receivers at the "limit of noise"). An experiment is not giving full information unless the metron-content of the observation exceeds that of the measurement.

(e) *Displacement of origin.*

An important point has not so far been raised. A measured value of a quantity x is inevitably represented as occupying an interval Δx . Accordingly since metron-content is essentially positive, the smallest "observable value" which we can justifiably attribute to a quantity *linearly related* to metron-content is not zero, but $\frac{1}{2}\Delta x$. The first interval which it can occupy on our conceptual scale has a width Δx . A detailed consideration of this point, and of the relation between different information scales, will, however, require a separate paper.

(f) *Summary.*

The suggestion here made may now be summarized as follows: full justice is done to the accuracy (the logical content) of any single measurement when it is described in terms of its *scale-unit* and its *metron-content*. The latter may be represented as the number of intervals, on an abstract conceptual "proper-scale", occupied by the measured quantity; but the number of intervals on this scale will not necessarily be proportional to the measured quantity, since the scale-unit may also depend on the latter. Quantities for which such a proportionality does hold, have a special claim to be regarded as fundamental in the particular context.

§ 5. STRUCTURAL INFORMATION.

(a) *Logon-content.*

It was Kant who said: "reason has insight only into that which it produces after a plan of its own." The design of an experiment is essentially the specification *a priori* of a pattern, of categories in terms of which alone the result can be described. All the events of the experiment must find a place in one or other of these, though of course not all categories will necessarily find an exemplar in a given experiment.

Since each independent category enables us to introduce a measure of differentiation—*i. e.* of form or structure—into our account of a result, we can regard knowledge thereof as providing us with *prior* or *structural information*. We can therefore define a unit of structural information or (using Gabor's term (*loc. cit.*)) a *logon*, as that which enables us to *formulate* one independent proposition, describing one independent feature of the result. (Whether when we have formulated the proposition we shall find any metrical information to give it logical content, is another matter.) The amount of structural information in a result, the *logon-content*, is thus the number of independent categories or degrees of freedom precisely definable in its description.

(b) *Logon-capacity.*

In many cases structure is defined in terms of a reference-coordinate. For example the density-pattern on a photographic plate can be described by a function of one or more space-coordinates; and the structure of a telephony signal can be specified by a time function. The *logon-capacity* of an experimental method can in such cases be defined as the number of logons which it specifies per unit of coordinate-interval, or coordinate-space if several coordinates are involved. The total number of independent categories or features in the result is then the integral of the logon-capacity over the extent of coordinate-tract occupied.

Thus the logon-capacity of a microscope in a particular region in the focal plane can be defined in *logons/cm.²*, and measures the resolving-power in that region; for suitable test-objects a resolving power in a given direction can also be defined, in *logons/cm.* The logon-capacity of a galvanometer or a communication-channel is measured in logons per second, and represents the number of (practically) independent readings per second which can be made with the apparatus.

(c) *Frequency-response.*

It is useful here to introduce the idea of frequency-response in a somewhat generalized form. With instruments measuring functions of time it is common practice to define performance in terms of their frequency-response to sinusoidal inputs of different time-periodicities. In the same way one can define Fourier-variables associated with other coordinates, in order to assess the performance of other types of instrument.

For example one could use sinusoidal density-patterns of different *space*-periodicities to define the "frequency-response" of a microscope, if one may so extend the use of the term. In each case a precise meaning can be given to the *frequency-bandwidth* of an instrument, as the effective range of input-frequencies (*i. e.* of the Fourier-variable) to which it is sensitive. (For example in a microscope this is a space-frequency range normally extending from zero to a value directly proportional to the aperture used. It thus measures the fineness of detail perceptible, in terms of a Fourier-type of analysis of the transparency-function which represents the object. One need hardly say that "frequency" here is quite distinct from the frequency of the light used.) It should be noted that with more than one coordinate the Fourier-variables must be represented in a Fourier-space in which "bandwidth" is defined by a "volume" not necessarily rectangular.*

(d) *Bandwidth and logon-capacity.*

The bandwidth of an instrument in the above sense is directly related to its logon-capacity. The relation arises from the well-known uncertainty-principle which can be written

$$\Delta f \cdot \Delta q \geq K_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where Δf represents the effective range of frequencies (conjugate to a coordinate q) to which the apparatus is sensitive, Δq twice the "uncertainty"† in q , and K_s a number which we may take to have the value $\frac{1}{2}$.

Equation (2) will later be justified from our present point of view. Meanwhile we may note that points on the q -axis cannot be defined uniquely at closer intervals than $K_s/\Delta f$, so that the logon-capacity is $\Delta f/K_s$ or $2\Delta f$. To attempt to talk of "an interval smaller than Δq " would be to try to construct a logical pattern identical with that of "a frequency higher than Δf " which cannot by definition appear in any result and is therefore observationally meaningless. The logon-content l of an experiment involving a tract of extent q is thus

$$l \leq q \cdot \Delta f / K_s \leq 2q \cdot \Delta f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

§ 6. REPRESENTATION OF INFORMATION.

(a) *The Information Matrix.*

The relation between the two features of information becomes clear in terms of a simple formalism. The definition of information with which we began was *operational*; "information" acquires meaning in terms of what it *does*. Thus to represent the total information, metrical and structural, provided by an experiment, we require a concise representation

* The author is indebted to a correspondent for raising this point.

† *i. e.* Δq is the effective range of q .

enabling us to deduce its effect on all conceivable statements relating to the result—in mathematical jargon, an *information-operator*. Suppose that an experiment has yielded l logons. To represent these as independent, we may think of them as defining a set of orthogonal rays in an l -dimensional “information-space”, with corresponding unit-vectors $\alpha_1 \alpha_2 \dots \alpha_l$. Taking these unit-vectors as a basis, we can now represent the total information yielded by an experiment by an *information-matrix* in the “canonical” form :

$$I = \begin{bmatrix} i_1 & \dots & \\ & i_2 & \dots \\ & & \dots & i_l \end{bmatrix}$$

where $i_1 \dots i_l$ are the metron-contents of the logons represented by $\alpha_1 \dots \alpha_l$ respectively.

Now relative to the logon-basis α_r , any proposed dependent statement can be represented by a unit vector-function ϕ , say, defined by $\phi = \sum_{r=1}^l f_r \cdot \alpha_r$, where the f_r are direction-cosines of ϕ (or its corresponding ray) and measure the relevance of the corresponding logons to the statement. If now for example we wish to calculate the total metron-content i_ϕ which the proposed statement would have, we need only form the product

$$i_\phi = \phi' \cdot I \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where ϕ' is the transpose of ϕ .

This procedure is common in the statistical estimation of significance, and has its analogue in the calculation of expectation values in quantum mechanics. In fact all the necessary mathematics at this point is at hand in textbooks on quantum theory (see *e. g.* Tolman 1938).

Without at present going into detail, we may note briefly three further points: (a) the *trace* of I , $i = \sum_{r=1}^l i_r$, represents the total metron-content of the original result; (b) a logically equivalent re-statement of the result corresponds to a *unitary transformation* of the basis, under which I will in general lose its simple diagonal form. This means that each “component” of a dependent statement is no longer a function of just one of the original logons. But (c) the trace of the information matrix is invariant under unitary transformations, so that—as our axiom demands—the total metron-content is unaltered by logical reformulation.

(b) Geometrical Representation.

As in the analogous formalism of quantum theory, this procedure can be given a simple geometrical representation (see *e. g.* Rojansky 1946). Because of its diagonal form, the operator I can be defined by a vector with components $\sqrt{i_1}, \sqrt{i_2}, \dots, \sqrt{i_l}$, which we can call the *information-vector*. This has a length $\sqrt{i}$, and defines a point,

or rather a volume-element, in the information space, relative to the origin. The information-vector, or the element whose occupance it specifies, can be regarded as the basic invariant behind the various statements we may make about the experiment; it is the "real result". The components of the vector along each axis are the square roots of the metron-contents of the corresponding logons, in accordance with our conservation-axiom, § 2 (b). Any dependent statement is now defined by a *direction* in the space, and its *metron-content* is the square of the projection of $\sqrt{i}$ in that direction. Unitary transformations correspond to *rotations* of the axes, and the significance of points (a) to (c) above is easily seen.

We have seen that the dimensional multiplicity l is fixed by the choice of experimental method (including the volume of space-time occupied). Performance of the experiment results fundamentally in the collection, and allocation to the various logons, of the metron-flow arising from the impact of data on the apparatus plus observer.

The quantal character of this process has one important effect. Since the number of ways in which i metrons can be distributed among l logons is limited, only a certain number of different results can conceivably be given by a particular experiment. An experiment, indeed, is an attempt to choose between a *finite* number of possibilities. If it were not so experimentation would be impossible.

The merit of an experimental method can therefore be gauged, in part at least, by the speed and ease with which it enables one to identify the appropriate information point.

We shall return to the question of the *detail* in a result in Part II., (§ 9).

II. Practical Details.

In developing the formalism of Part I. we incurred a number of debts and shelved several problems. These we must seek now to remove in a more detailed discussion.

§ 7. METRICAL SCALE UNITS.

The first omission concerned the nature of the limitations on the accuracy of statements. In either metrical or structural propositions, a numerical magnitude must be quantized in order to be logically representable. The origin of the quantization in the two cases is quite different however, as we shall now show.

(a) *The metrical information-axiom.*

A single metrical statement about a quantity, y say, is logically sterile unless we identify it by a coordinate-label, q say. The proposition: "I have received i metrons relating to y " represents our actual experience, and has no vagueness; but vagueness arises directly we conceive y to be a function of q , and try to formulate a proposition about y in terms of q .

quantities, the sum *Entropy-plus-Information-content* would rank as a fundamental invariant. (See § 10 (c) below). From this viewpoint, temperature would appear as a measure of the scale on which energy is represented in the information-pattern.

An analogous relation holds in the case of a space-statistic such as the density of a photographic image.

§ 8. STRUCTURALLY-DEFINED SCALE-UNITS.

(a) *The structural Information-axiom.*

In structural propositions the absolute magnitude of y is irrelevant. They are essentially definitions of propositional functions of which y is to be the argument. Accordingly the scale-unit of q can be defined only in terms of coincidence-relations independent of the magnitude of y . In § 5 we used the concept of bandwidth (generalized) to define a scale-unit Δq which was a property of the apparatus used, ascertainable beforehand by an independent experiment, and hence counting as prior information on subsequent occasions. The general relation (2) which we used then, can be justified directly in terms of our initial axiom that only coincidence-relations or compounds thereof are valid as logical elements in scientific statements.

Let us consider then the case of a simple harmonic function of q , with periodicity f . It will be associated with definitive points on the q -axis, independent of amplitude, whenever the function crosses the axis, *i. e.* at intervals of half a period. (It is also arguable that ingenious experimentation should make it possible to define the quarter-period or the radian-period independently of amplitude.)

In any case, the scale of q is conceptually provided with a set of points at intervals of K_s/f , where K_s is of the order 1.

These points, however, are all logically identical, representing collectively a single fact—the accurate value of f . What we want is a set of uniquely identifiable points, to serve as labels. To single out desired points we must provide a comparison-pattern to act as a “pointer”. For instance a frequency $f - \Delta f$ will produce a pattern coinciding with the first at intervals of $K_s/\Delta f$. If all values of Δf from zero can be observed, a continuous range of intervals from infinity down to $K_s/\Delta f$ can be observationally defined. The *structural scale-unit* of q , Δq , is thus $K_s/\Delta f$. Conversely, if Δq is an arbitrary interval, the number l of logons relating to it which can be formulated can be written as $l \leq \Delta q \cdot \Delta f / K_s$, since l is integral. Thus for a single logon $\Delta q \cdot \Delta f \geq K_s$ (equation (2)). (It should be remembered that in practice the terms “effective length” and “bandwidth” often have a slightly different numerical connotation from that here employed.)

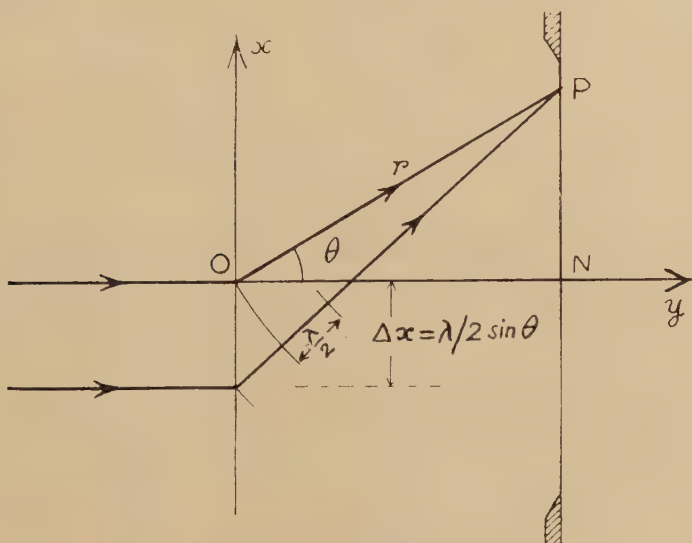
It is of interest that the metron-density function specified by a single logon, for which equation (2) becomes an equality, is a Gaussian probability-function, since our data are exactly the assumptions made

by Gauss in his derivation. (See Appendix.) Analysis of a function into logons is effectively description in terms of superimposed Gaussian functions (Gabor, *loc. cit.*).

(b) *Resolving Power.*

To illustrate these ideas we may examine the problem of optical resolving power. As our purpose is only expository we shall consider the somewhat unrealistic case of a narrow rectangular aperture and an ideal object which can be represented by a transparency-function of only one coordinate, x say, parallel to the long edge of the aperture. The situation can then be represented by the two-dimensional diagram of fig. 1. P is a point on the aperture, with coordinates (r, θ) relative

Fig. 1.



Normal incidence.

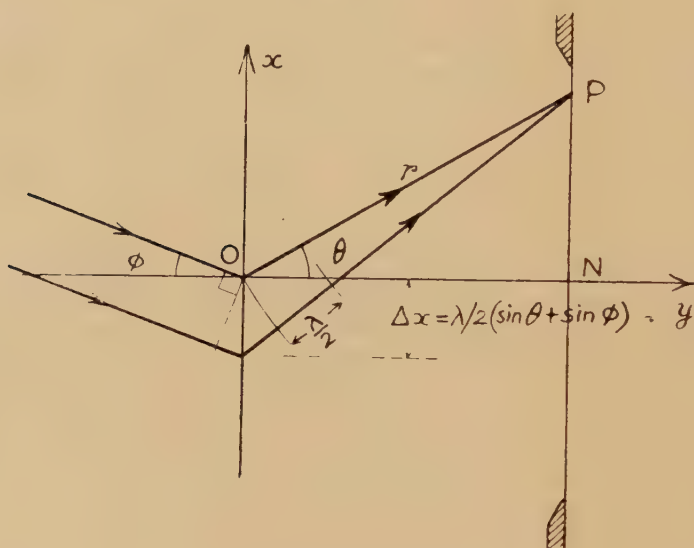
to a point O on the object. Light is incident in the direction Oy . The problem has two parts. Firstly, we shall establish a relation between the position of P and the information gained through it about the object ; and secondly, we can calculate the logon-capacity for a given type of aperture.

Suppose that light of wavelength λ is received through P. This makes it possible to distinguish logically points along the possible light paths to P at intervals of $\lambda/2$, so that with centre P, we can give logical significance to a series of arcs of radii $n\lambda/2$, where n has integral values. These arcs will intersect the x -axis and establish coincidence-relations with the incident wavefront at intervals of $\lambda/2 \sin \theta$, in the vicinity of O.

There is thus a correspondence between the point P and a space-function having a periodicity $f_x = (\sin \theta)/\lambda$ in the object. Indeed if the aperture is closed everywhere except at P and the pole, N, the pattern seen has just this periodicity.

This analysis has assumed that a plane wavefront was incident normally on the x -axis. If instead the light is incident in the xy plane at an angle ϕ to OX (fig. 2) the possible light-paths from P must be traced through the object to their origin on the incident wavefront; and the logically describable paths will be those whose *total* lengths are integral multiples of $\lambda/2$. The corresponding space-periodicity defined on the x -axis will then be $f_x = \left(\frac{1}{\lambda}\right)(\sin \theta + \sin \phi)$.

Fig. 2.

Incidence at angle ϕ .

To calculate the logon-capacity we need only observe that the "bandwidth" is directly proportional to the range of $\sin \theta$. Thus if θ can have all values between $\pm \theta_m$, $\Delta f = \left(\frac{2}{\lambda}\right)(\sin \theta_m)$, and the logon-capacity is $(4/\lambda)(\sin \theta_m)$ logons per cm.

This logon-capacity will not, however, be realized in the case of normal incidence, for then f_x is only $(1/\lambda) \sin \theta_m$, and only $(2/\lambda) \sin \theta_m$ points are specified on the x -axis. Only by increasing ϕ to θ_m can the full resolving-power be achieved. Increasing ϕ beyond this value, to give so-called "dark-ground illumination", will increase f_x but not Δf . The number of logons per cm. will not be increased, though owing to the removal of background light the visibility of detail is generally improved.

These results are well-known, but they have here been deduced by way of illustration from an axiom which can perhaps be generalized as follows : *the limit to scientific observation is the limit of our logical vocabulary.* If a phenomenon can be *defined* (in terms of the atomic propositions of the scientific method) it can in principle be *observed*.

§ 9. ESTIMATION OF IMPORTANCE.

(a) Variety.

We can now calculate the number of logically definable information-points in the l -space of § 6. The number, which we may call M^* , measures the total range or variety of the possible results, out of which the experiment has chosen one. It is therefore a measure of the *importance* of a result. For a given upper limit to i , M will depend on the way in which quantization occurs. Assuming that the apparatus is such that all integral values of each i_r can be distinguished, M is easily shown to be

$$M = (i+l) ! / i ! l ! \quad (10)$$

if the total metron-content can have all integral values from 0 to i . If, however, i must always have its maximum value, M is found to be

$$(i+l-1) ! / i ! (l-1) ! \quad (11)$$

(This is reasonable, since the possibility of varying i merely confers one further degree of freedom on the system as a whole.)

We can also distinguish between cases in which (a) the apparatus sets no limit to the metron-density ρ_i , but only to the total i , and (b) the apparatus sets an upper limit to ρ_i . In case (a), any i_r can have the full value i , and formula (10) applies. The maximum of M is attained when $l=i$, so that the optimum metron-density in a statement is one metron per logon, as we have seen before. If i is large, Stirling's formula for $i !$ then gives

$$M \simeq 2^{2i} / \sqrt{(\pi i)}. \quad (12)$$

In case (b), each i_r can equal but not exceed $\rho_i \cdot \Delta q$. Interest now focuses on the value of M per unit of q , M_q , which is given by the number of unit " volumes " in a l -dimensional " cube " measuring $(1+\rho_i \cdot \Delta q)$ in each direction. (The addition of 1 allows for the value $i_r=0$). Since $l=1/\Delta q$ per unit of q , this makes

$$M_q = (1+\rho_i/l)^l. \quad (13)$$

In the useful limit $l=\rho_i$, so that

$$M_q = 2^l. \quad (14)$$

* The notation here has been chosen so as to coincide as far as possible with that of Shannon (1949).

(b) Detail.

A convenient measure of the *detail* in a result is the equivalent number of "yes-no relations",

$$N = \log_2 M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)^*$$

For equation (12), in case (a), N has the value

$$N = 2i - \frac{1}{2} \log_2 (\pi i) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12a)$$

in this case approximately twice the logon- or metron-content. The ability to group all metrons under any one logon approximately doubles the number of independent two-valued propositions which can result from the experiment. In case (b), equation (14), the density of detail

$$C = \log_2 M_q = l, \text{ or } \Delta f / K_s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14a)$$

Note, however, that even increasing l to infinity only makes $M_q \rightarrow e^q$, and

$$C \rightarrow \rho_i \log_2 e. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

In some cases (*e.g.* where scale readings are proportional to voltage, in the presence of a constant noise voltage), only whole-number *coordinates* of the information point may be distinguished (and not merely whole-number combinations of metron-content). M_q then cannot exceed

$$(\sqrt{1 + \rho_i/l})^l, \quad (\text{cf. (13)}) \quad \text{or} \quad M_q \leq \left(1 + \frac{\rho_i}{l}\right) l/2. \quad . \quad . \quad . \quad (17)$$

This is the number of unit-spheres in an l -dimensional sphere of radius $\sqrt{1 + \rho_i/l}$. When $l = \rho_i$,

$$M_q = 2^{l/2}, \quad \text{and} \quad C = l/2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

(c) Increase of detail.

These equations lead to one very practical point. To increase the amount of detail in a result, it is more profitable to increase l than i , unless already $l > i$; but *both methods can be used*.

In other words we have an incentive to apply ingenuity in such matters as the resolving-power of optical instruments, knowing that as long as the metrical information provided is not usefully employed, something practical remains to be done, even if classical theory appears to be discouraging. In the field of communication this aspect is already under vigorous development.

The fundamental point is that as long as ρ_i exceeds the logon-capacity l , there exists a metrical scale-unit smaller than the structural one. One may, therefore, legitimately aspire to formulate in some way or other more than l independent propositions about the structure in question.

* This is analogous to Shannon's measure of "information", and the relations here derived may be compared with those in his paper on *Communication in the presence of Noise* (1949).

§ 10. PRACTICAL ASPECTS.

Thinking in these terms proves helpful and often stimulating in assessing the shortcomings, merits and possibilities of different experimental methods. In particular, subjective preferences in matters of debate can often be usefully supplemented by quantitative considerations of the kind suggested here.

(a) *Precision.*

The precision of a single measurement can be enhanced indefinitely by increasing the space-time tract *irreducibly** associated with it. Thus in experiments to determine a constant, efforts should be directed towards "logon-compression"—reducing the frequency-response (*i. e.* the logon-capacity) of the apparatus, with respect to time and space. In short best results are obtained by acting consistently with one's belief that the constant will not alter with time or position, so that one logon will be sufficient. In a sequence of operations, the logon-capacity of each should be adjusted so that the metron-content does not greatly exceed the value which it has in the stage with the narrowest bandwidth. This will enable each subsidiary operation to occupy the minimum space and time, so giving a higher overall metron-capacity, and making possible more repetitions of the experiment in a given space-time tract.

(b) *Structural detail.*

In experiments to determine structure on the other hand, the aim should be to reduce the relevant conceptual unit of coordinate to the minimum at which *metron-content* is adequate. One may even say (with reservations) that there is almost always an improvement in resolving power (with a given input of energy) when intelligent steps are taken to sacrifice metrical information. Any device which makes it possible to define a smaller interval can in principle be used to enhance the detail attainable.

(c) *Conservation of Information.*

Normally a compromise is required, and the formalism of § 6 is illuminating. Processes of logon-compression and expansion can be represented by rotations of the information-vector. In most processes some attenuation will occur, if they are not reversible in the thermodynamic sense. On the other hand if they are, we deduce from the invariance of entropy plus metron-content that a principle of *conservation of information* applies: *It is impossible for a reversible system to destroy information, however complex may be the process required to reconstitute it in its original form.* At the same time *no artifice of design or manipulation of a result can increase the accuracy of a measurement beyond the limit set by i* (*i. e.* an increase in the metron-content of individual logons can be

* See § 7(b).

bought at the expense of logon-capacity, but the limit is set by the total quantity i , which we have seen to depend on the expanse of coordinate-tract devoted to the experiment.

This in effect is the general principle which lies behind the many "incompatibilities" encountered in experimental design.

(d) *Illustrations.*

A typical example of the benefits of logon-compression is the information provided by the analyses of insurance companies. They reduce erratic data obtained in enormous detail, to a small set of propositions of great reliability. An analogous process occurs when a large capacitor is used to smooth the output of a photoelectric cell.

The converse process is well exemplified by the "equalization" of a telephony signal distorted by a narrow-bandwidth channel. Here the effective dimensionality of the information-vector is *increased*. The information-vector is rotated into some of the "dormant" dimensions, which were occupied by larger components in the original signal but had them attenuated in the channel. The consequent all-round reduction in information per logon appears in practice as a corresponding increase in noise.

If, however, the channel is electrically non-dissipative (*i. e.* contains only reactive components), the original signal can theoretically be restored without loss of information.

As a final illustration of the kind of approach here suggested, we may consider the type of aperture in a microscope which would give the maximum of contrast from a particular object. "Contrast" is essentially a matter of metron-content, so that the question may be put: "What transparency-function in the aperture will yield the maximum metron-content?" We may for illustration make the gross simplification of assuming that contrast is impaired only by random background-light uniformly distributed over the aperture. Again considering the linear case for simplicity, we can describe the object by its "space-spectrum" $S(f_x)$, which will give rise to a corresponding intensity function $S(\xi)$ over the aperture, which extends from ξ_m to $-\xi_m$. If the transparency-function is $t(\xi)$, the total random background power received W_B is proportional to $\int_{-\xi_m}^{\xi} t(\xi) d\xi$, while the total coherent power W_C is $\int_{-\xi_m}^{\xi} S(\xi)t(\xi) d\xi$.

The ratio $\frac{W_C}{W_B}$ will be a maximum when $t(\xi) \equiv S(\xi)$ apart from a constant factor. Thus the aperture should be masked by a transparency representing the spectrum of the expected image. If the requirement is for the maximum resolving-power and metron-content, it is known (see Appendix) that the spectrum corresponding to one logon is Gaussian in outline. Thus the best compromise will be obtained for general purposes by masking the aperture with a Gaussian transparency-function.

§ 11. CONCLUSION.

The chief aim of the present paper has been to present a new way of thinking of experimentation, an effort to isolate the abstract concept which represents the real currency of scientific intercourse, from the various contexts in which it appears. Normal modes of expression of scientific facts involve logical redundancy in vocabulary. Logical analysis in terms of irreducible elements leads to a minimum vocabulary, in which a fact is represented by a conjunction of its abstract information-content with the particular context which gives it concrete meaning.

Towards this minimum vocabulary the present paper can claim to make only a preliminary and naive contribution. It has been seen, however, that scientific information is inherently quantal in its communicable aspects; and that the various uncertainty-relations of physics, though arising in different ways, are basically expressions of this one fact. *Analogies* in physics emerge as *identities* between basic structures, and the validity of arguments from analogy is easily assessed. It is to be expected that problems analysed in these terms would find an answer with the minimum of logical manipulation.

The concepts here introduced find ready application in the experimental field. Their fruitfulness in other connections remains to be seen. One cannot help feeling, for example, that the dual character of communicable information has a direct bearing on the physiology of thought-processes and the design of electronic reasoning-instruments. Its metrical aspect in particular suggests a possible "analogue" mechanism which is not at present receiving as much attention as the "digital" one, and which it is also hoped to examine in a later paper.

The philosophical reader will recognize that the discussion has not centred on the nature of *reality*, but on the nature of the propositions which the scientific method permits us to make about reality. The implications which this "compulsion by the logical form" have for fundamental theory cannot be examined here; but if the present approach can help in any way to close familiar loopholes in otherwise attractive epistemological arguments, it will repay further study.

§ 12. ACKNOWLEDGMENTS.

This paper is an expanded version of a lecture given to the Maxwell Society of King's College in January 1948. Since then several papers have appeared, chiefly in American journals, dealing with communication theory in terms of a formalism similar in some respects to that suggested here. This has entailed some duplication in § 9 of the present paper, but it has still seemed worth while to present derivations of the various relations in their original form, since the approach is here somewhat different and more general. In particular it is hoped that controversy over alleged "refutations" of particular definitions (see *e.g.* Tuller 1949) may be allayed by the clarification of the complementary relation between different senses of "Information".

It is a pleasure to express indebtedness to Dr. D. Gabor, both for contributions which his work has made to the thought of the present paper, and for helpful comments on its presentation. Especial thanks are due to the many colleagues of the author whose detailed criticism has helped to prevent a number of errors and obscurities.

§ 13. GLOSSARY OF TERMS USED.

Bandwidth.—In general terms, the region of Fourier-space to which the output of an instrument is confined. In particular, the effective frequency-range (conjugate to a given coordinate) to which it responds (§ 5(c)).

Information.—That which alters the total statement representing “all that is the case” (§ 2).

Information-space.—The space in which independent propositions are represented by orthogonal rays, and their metron-contents by the squares of distances along these rays (§ 6).

Information-vector.—The vector whose components in information-space are the distances just mentioned (§ 6).

Logon.—Unit of structural information, specifying *form* (§ 5(a)).

Logon-capacity.—Number of logons per unit of coordinate-space (§ 5(b)).

Logon-content.—Number of logons in a statement (§ 5(a)).

Metron.—Unit of metrical information, specifying *magnitude* (§ 4(b)).

Metron-capacity (cf. logon-capacity) (§ 7(c)).

Metron-content (cf. logon-content).—A measure of the *support* given by the data to a statement (§ 4(b)).

Natural unit.—The coordinate-interval over which one metron is acquired (§ 7(b)).

Proper-scale.—A representational scale on which *magnitude* is represented by a *number* of intervals occupied, though not necessarily linearly related to that number (§ 4(c)).

Scale-unit.—The “smallest meaningful interval” on a scale (§ 4(a); §§ 7 and 8).

APPENDIX.

The metron-density function specified by a logon.

Gauss based his derivation of the probability function on three assumptions :

(A) The probability of committing an error z in a magnitude X depends only on z .

(B) All values of X are *a priori* equally acceptable ; the probability is not affected by the value of X .

(C) The most probable value of a magnitude X of which we possess n readings, is the arithmetic mean of these readings.

From these it is shown in any text book on probability that the probability of committing an error of between z and $z+dz$ is $h/\sqrt{(\pi)} \exp(-h^2z^2)dz$, where h is a constant denoting the accuracy of the measurement.

Our problem is to derive a function with analogous properties :

(a) The metron-density specified by a logon relating to q_0 depends only on the displacement q from q_0 .

(b) The density-function is not affected by the choice of q_0 .

(c) The function has a single maximum at the centroid of the metron distribution.

This follows from the facts that : (1) the two possible elementary occupance-relations specified by the logon must each have a probability of $\frac{1}{2}$. There must therefore be as many metrons in support of occupance of the region to the left of q_0 , as to that of the right of q_0 . (2) Only one *singularity* can be defined by one logon, so that both asymmetry and multiplicity of turning-values is ruled out. (3) The singularity cannot be a minimum, since one-half of the total area under the curve (metron-content) lies between finite limits defined as $\pm \Delta q/2$. We are thus justified in making use of Gauss's result and deducing that the density function is of the form $A \exp(-h^2q^2)$. If we use condition (c) (3), that the inter-quartile range is to be Δq , we find from tables that h must be $0.941/\Delta q$.

The Fourier transform of this function is also Gaussian, the indices h in the two cases being reciprocal. The frequency spectrum corresponding to a single logon has the same shape as the metron-density function, and could have been derived in the same way.

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XXV. *Notices of New Books and Periodicals received.*

The Strength of Plastics and Glass. By R. N. HAWARD. [Pp. 245.] (London : Cleaver-Hume Press Ltd., 1949). Price 30s. net.

THE scope of this monograph is satisfactorily indicated by its title. The emphasis is mainly on experimental results : indeed in the present state of the theory this could hardly be otherwise. It is primarily as a source of factual information that the book will be valuable, with its fifty tables of data, hundred diagrams, and nearly four hundred references to recent literature. Those not primarily concerned with the technology of the subject will find it a comprehensive summary of the present state of knowledge in a rapidly developing field of investigation. N. T.

Colloid Science, edited by H. R. KRUYT. Volume 2. *Reversible Systems.* (Elsevier Publishing Company, Ltd. 1949). [Pp. 753.] Price £4 10s.

DURING the past three decades Professor Kruyt and his school at Utrecht have made contributions of the greatest importance to *Colloid Science*. These contributions have ranged over the whole field of stability of lyophiles, lyophobic, and emulsions, and it is therefore with great pleasure that we welcome this, the first to appear of two large volumes covering many of those topics with which the Dutch school has been concerned. Professor Kruyt is careful to point out in his preface that the book is not comprehensive, but the wide range of topics and the authoritative nature of the treatments by the various expert contributors will make this book essential for all chemistry libraries, and many chemists will want to possess their own copy, in spite of the very high price.

There are seven contributors to this volume. Bungenburg de Jong writes the first chapter, which deals with definitions and general matters like the applicability of the Phase Rule to colloid systems. Modern work on polymers has necessitated a review of our use of many classical terms. De Jong has six more chapters dealing with macromolecular electrolytes, coacervates and complex colloids, thus covering a whole range of water-soluble systems. While the treatment of this section is perhaps rather too extensive in comparison with the others, an accessible account in English has been needed for some time. Also, water-soluble polymers have usually received scant treatment in books on polymer chemistry. Houwink and J. J. and P. H. Hermans are responsible for a very well-documented treatment of non-electrolytic polymers. A strange absentee from the section on osmotic pressure is the well-known apparatus of Fuoss and Mead, which, in fact, is much used by polymer chemists. However, the treatment is otherwise very complete, particularly on the theoretical side. A good account is given of the fundamental theoretical work of Huggins and Flory, and the key position occupied by the experimental work of Gee and Treloar in this field is clearly to be seen.

The book has been very well translated from the Dutch by Dr. L. C. Jackson. It should make a useful companion to the recent volumes under the same title by A. E. Alexander and P. Johnson, to which it is complementary in many respects. The type, illustrations and binding are excellent. We may expect that this timely and thorough account of certain aspects of *Colloid Science*, will greatly stimulate research in the subject. D. D. E.

[The Editors do not hold themselves responsible for the views expressed by their correspondents.]